

Optimal Portfolio under Fractional Stochastic Environment

Jean-Pierre Fouque*

Ruimeng Hu†

March 22, 2017

Abstract

Rough stochastic volatility models have attracted a lot of attentions recently, in particular for the linear option pricing problem. In this paper, starting with power utilities, we propose to use a *martingale distortion representation* of the optimal value function for the nonlinear asset allocation problem in a (non-Markovian) fractional stochastic environment (for all Hurst index $H \in (0, 1)$). We rigorously establish a first order approximation of the optimal value, where the return and volatility of the underlying asset are functions of a stationary slowly varying fractional Ornstein-Uhlenbeck process. We prove that this approximation can be also generated by a fixed zeroth order trading strategy providing an explicit strategy which is asymptotically optimal in all admissible controls. Furthermore, we extend the discussion to general utility functions, and obtain the asymptotic optimality of this fixed strategy in a specific family of admissible strategies.

Keywords: Optimal portfolio, Fractional stochastic processes, Martingale distortion, Asymptotic optimality.

1 Introduction

In this paper, we study the Merton problem under a non-Markovian fractional stochastic environment, and we are able to provide an explicit trading strategy which is asymptotically optimal in the case of power utilities and asymptotically optimal in a specific family of general utilities.

The portfolio optimization problem was first studied in the continuous-time framework by Merton [1969, 1971], where risky assets are considered following the Black-Scholes-Merton model with constant returns and constant volatilities. Under this setup, Merton provided explicit solutions on how to trade stocks and/or how to consume so as to maximize one's utility, when the utility function is of specific types, for instance, Constant Relative Risk Aversion (CRRA). After these seminal papers, the optimal portfolio and consumption problem has been extensively studied in the financial market subject to imperfections. For instance, Cox and Huang [1989] and Karatzas et al. [1987] studied the case of incomplete market; transaction cost has been considered by Magill and Constantinides [1976] and a user's guide by Guasoni and Muhle-Karbe [2013]; investment under portfolio constraint are studied by Grossman and Zhou [1993], Cvitanic and Karatzas [1995] and Elie and Touzi [2008], just to name a few.

A key factor in Merton problem is the modeling of underlying assets, and empirical studies suggest that volatility is stochastic. In this direction, we refer the readers to Zariphopoulou [1999] for the case of non-linear local volatility models, Chacko and Viceira [2005] for the case of a particular Heston-like stochastic volatility model, Lorig and Sircar [2016] for the case of local-stochastic volatility, and Kramkov and Schachermayer [2003] for the case of general analysis for semimartingale models, to list a few.

Most of the work have focused on the Markovian models of the volatility. However, in a recent series of papers, non-Markovian structured models seem to better describe the data, especially short-range dependence. In Gatheral et al. [2014], it is beautifully demonstrated that stochastic volatility driven by a fractional Brownian motion (fBm) with Hurst coefficient $H < \frac{1}{2}$, so-called *rough fractional stochastic volatility* (RFSV), is essentially relevant to observed data. Jaisson and Rosenbaum [2016] and Omar et al. [2016] showed that RFSV is a natural scaling limit of a general model of Limit Order Book (LOB) based on Hawkes processes.

*Department of Statistics & Applied Probability, University of California, Santa Barbara, CA 93106-3110, fouque@pstat.ucsb.edu. Work supported by NSF grant DMS-1409434.

†Department of Statistics & Applied Probability, University of California, Santa Barbara, CA 93106-3110, hu@pstat.ucsb.edu.

Meanwhile, multi-scale factor models for risky assets were considered in the portfolio optimization problem in Fouque et al. [2016] and Fouque and Hu [2016b], where return and volatility are driven by a fast mean-reverting factor and a slowly varying factor. Specifically, Fouque et al. [2016] heuristically provided the asymptotic approximation to the value function and the optimal strategy for general utility functions, by analyzing a non-linear Hamilton-Jacobi-Bellman partial differential equation (HJB PDE).

In this paper, we shall consider both the scales and non-Markovian structure for modeling the underlying assets. As in Fouque and Hu [2016a], and in particular because of the relevance for long-term investments, we only consider one *slowly varying fractional stochastic factor* denoted by $Z_t^{\delta,H}$ for $0 < H < 1$. The cases with fast mean-reverting as well as multi-scale models, limited to $H > 1/2$, are studied in the paper in preparation Fouque and Hu [2017]. As in Garnier and Solna [2015], we model $Z_t^{\delta,H}$ by a fractional Ornstein-Uhlenbeck (fOU) process, which satisfies the following stochastic differential equation (SDE)

$$dZ_t^{\delta,H} = -\delta a Z_t^{\delta,H} dt + \delta^H dW_t^{(H)},$$

where δ is a small parameter, and $W_t^{(H)}$ is a fractional Brownian motion with Hurst index H . We refer to Section 3.1 for a brief introduction to fBm and fOU, and to Mandelbrot and Van Ness [1968], Cheridito et al. [2003], Coutin [2007], Biagini et al. [2008], Kaarakka and Salminen [2011] for more details.

Pricing options under such RFSV models is indeed a challenge since the model is non-Markovian and PDE tools are no longer available. However, when the fractional stochastic volatility factor is slowly varying (small δ), one can obtain a practical approximation using the so-called “epsilon-martingale decomposition” method designed in Fouque et al. [2000] and Fouque et al. [2001]. This has been recently carried out for slowly varying RFSV models in Garnier and Solna [2015] where a correction to Black-Scholes formula for fractional SV is obtained. Note that the problem is non-Markovian but remains linear in the case of option pricing.

Main results. In this paper, we study the nonlinear terminal utility maximization problem under the RFSV model (3.10). For power utilities, by a martingale distortion representation, we rigorously obtain an expression for the value process at any time and for all $H \in (0, 1)$, as well as an expression for the corresponding optimal portfolio. In the regime of small δ , these expressions take the form of a leading order term plus a first order correction of order δ^H . This is done by expanding the martingale distortion representation around a “frozen” volatility at the observed value $Z_0^{\delta,H}$ at time $t = 0$. For H relatively small, close to 0.1 as demonstrated in Gatheral et al. [2014], the first order correction of the value process is relatively large, and should also be generated by any good practical strategy. Our result nicely shows that the leading order of the optimal strategy, which is explicit in terms of the underlying asset and the current factor level, therefore easily implemented, will generate the value function up to order δ^H , that is including the first correction. In other words, the δ^H term in the expression of the optimal strategy is not needed to give such correction to the value process. However, it is given explicitly and can be easily implemented to improve the strategy by taking into account inter-temporal hedging. For general utility functions, using the epsilon-martingale decomposition method and the properties of the risk tolerance function for the Merton problem with constant coefficient, we obtain an approximation for the portfolio value corresponding to a given strategy, and, as in Fouque and Hu [2016a] in the Markovian case, we show that this strategy is asymptotically optimal in a specific class of admissible strategies.

Organization of the paper. The rest of the paper is organized as follows. In Section 2, we present the martingale distortion transformation under general stochastic volatility models first derived in the Markovian case in Zariphopoulou [1999], and in non-Markovian settings in Tehranchi [2004]. Here the drift and volatility of the underlying asset are driven by a stochastic process which is not required to be Markovian nor a semimartingale. We also present a generalization to the multi-asset case. In Section 3, we derive the asymptotic results when the stochastic factor is fractional and slowly varying. The approximation to the value process and optimal portfolio are given in Section 3.3 and Section 3.4 respectively. It is also shown that the leading order of the optimal portfolio is optimal in the full class of admissible strategies up to δ^H . Merton problem with a general utility function is discussed and asymptotic optimality results are presented in Section 4. We conclude in Section 5.

2 Merton Problem with Power Utilities and Stochastic Environment

Denote by S_t the underlying asset price whose return and volatility are driven by a stochastic factor Y_t ,

$$dS_t = \mu(Y_t)S_t dt + \sigma(Y_t)S_t dW_t, \quad (2.1)$$

with assumptions on $\mu(y)$ and $\sigma(y)$ to be specified later. Here Y_t is a general stochastic process that is adapted to $\mathcal{G}_t := \sigma\{W_u^Y : 0 \leq u \leq t\}$, where W_t^Y is a Brownian motion generally correlated with the Brownian motion W_t driving the price S_t :

$$d\langle W_t, W_t^Y \rangle = \rho dt, \quad |\rho| < 1.$$

Also define \mathcal{F}_t as the natural filtration generated by (W_t, W_t^Y) .

Denote by π the investor's strategy and by X_t^π the corresponding wealth process. The quantity $\pi_t \in \mathcal{F}_t$ represents the amount of money invested in the risky asset at time t , while the rest $X_t^\pi - \pi_t$ earns a risk-free interest rate r (constant). Assuming that the strategy π is self-financing, and, without loss of generality, that the risk-free interest rate is zero, $r = 0$, the dynamics of the wealth process X_t^π is given by:

$$dX_t^\pi = \pi_t \mu(Y_t) dt + \pi_t \sigma(Y_t) dW_t. \quad (2.2)$$

The investor's goal is to find the optimal strategy so as to maximize her expected utility of terminal wealth. Mathematically, she aims at identifying the optimal value

$$V_t := \sup_{\pi \in \mathcal{A}_t} \mathbb{E}[U(X_T^\pi) | \mathcal{F}_t], \quad (2.3)$$

and the optimal strategy π^* . The set \mathcal{A}_t is the class of all admissible strategies:

$$\mathcal{A}_t := \{\pi_t \in \mathcal{F}_t : X_s^\pi \text{ in (2.2) stays nonnegative } \forall s \geq t, \text{ given } \mathcal{F}_t\}, \quad (2.4)$$

where zero is an absorbing state for X_t^π (bankruptcy), and $U(\cdot)$ is a utility function. Additional assumptions on \mathcal{A}_t will be described later. Specifically, $U(\cdot)$ will be of power type in this section and Section 3, and be a general utility function with additional assumptions in Section 4.

In order to motivate the martingale distortion transformation that we will introduce in Section 2.2, we first recall in the next subsection the distortion transformation obtained by Zariphopoulou [1999] in the Markovian case with power utility

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \gamma \neq 1. \quad (2.5)$$

In Section 2.3, we generalize to the multi-asset case.

2.1 The Distortion Transformation

In the Markovian setup, Y_t is a diffusion process following the stochastic differential equation of the form

$$dY_t = k(Y_t) dt + h(Y_t) dW_t^Y,$$

and the value function $V(t, x, y) := \sup_{\pi \in \mathcal{A}_t} \mathbb{E}[U(X_T^\pi) | X_t = x, Y_t = y]$ is a solution to the Hamilton-Jacobi-Bellman (HJB) equation given in Fouque et al. [2016]. The distortion transformation is given by

$$V(t, x, y) = \frac{x^{1-\gamma}}{1-\gamma} \Psi(t, y)^q, \quad (2.6)$$

with

$$q = \frac{\gamma}{\gamma + (1-\gamma)\rho^2} \quad (2.7)$$

which results in canceling $(\Psi_y)^2$ terms in the HJB equation. Consequently, Ψ solves the linear PDE

$$\Psi_t + \left(\frac{1}{2} h^2(y) \partial_{yy} + k(y) \partial_y + \frac{1-\gamma}{\gamma} \lambda(y) \rho h(y) \partial_y \right) \Psi + \frac{1-\gamma}{2q\gamma} \lambda^2(y) \Psi = 0, \quad \Psi(T, y) = 1,$$

where $\lambda(y)$ is the Sharpe ratio $\lambda(y) := \mu(y)/\sigma(y)$.

By Feynman-Kac formula, we observe that Ψ can be expressed as

$$\Psi(t, y) = \widetilde{\mathbb{E}} \left[e^{\frac{1-\gamma}{2q\gamma} \int_t^T \lambda^2(Y_s) ds} \middle| Y_t = y \right], \quad (2.8)$$

where under $\widetilde{\mathbb{P}}$, $\widetilde{W}_t^Y = W_t^Y - \int_0^t \rho \left(\frac{1-\gamma}{\gamma} \right) \lambda(Y_s) ds$ is a standard Brownian motion.

The formula in the next subsection generalizes (2.8) without using any PDE argument.

2.2 Martingale Distortion Transformation

The martingale distortion transformation is motivated by the formulas (2.6) and (2.8). It has been derived in Tehranchi [2004] with a slightly different utility function. For the sake of clarity, we restate it here, and we propose a short proof based on verification using stochastic calculus.

Note that in the following Proposition 2.3, Y_t is a general stochastic process adapted to \mathcal{G}_t , and *it does not need to be Markovian, nor a semimartingale*. In particular, in the Section 3, we will be able to apply it to the case that Y_t is a fractional process.

Assumption 2.1 (Power Utility). *The admissible strategies $\pi \in \mathcal{A}_t$ in (2.4) satisfy*

$$\mathbb{E} \left[\sup_{t \in [0, T]} (X_t^\pi)^{2p(1-\gamma)} \right] < +\infty, \text{ for some } p > 1, \quad \text{and} \quad \mathbb{E} \left[\int_0^T (X_t^\pi)^{-2\gamma} \pi_t^2 \sigma^2(Y_t) dt \right] < \infty.$$

Before introducing the assumptions on the processes (S_t, Y_t) , we need to define a new probability measure $\widetilde{\mathbb{P}}$ by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \exp \left\{ - \int_0^T a_s dW_s^Y - \frac{1}{2} \int_0^T a_s^2 ds \right\}, \quad (2.9)$$

where a_t is \mathcal{G}_t -adapted given by

$$a_t = -\rho \left(\frac{1-\gamma}{\gamma} \right) \lambda(Y_t), \quad (2.10)$$

so that under $\widetilde{\mathbb{P}}$, $\widetilde{W}_t^Y := W_t^Y + \int_0^t a_s ds$ is a standard Brownian motion. Under the following assumption, a_t is bounded.

Assumption 2.2. (i) *The SDE (2.1) for S_t has a unique strong solution. The function $\lambda(\cdot)$ is assumed to be bounded and $C^2(\mathbb{R})$, and functions $\lambda'(\cdot)$, and $\lambda''(\cdot)$ are at most polynomially growing.*

(ii) *Define the $\widetilde{\mathbb{P}}$ -martingale*

$$M_t = \widetilde{\mathbb{E}} \left[e^{\frac{1-\gamma}{2q\gamma} \int_0^T \lambda^2(Y_s) ds} \middle| \mathcal{G}_t \right], \quad (2.11)$$

then, one has

$$dM_t = M_t \xi_t d\widetilde{W}_t^Y, \quad (2.12)$$

by the Martingale Representation Theorem. We require $\xi_t \in L^{2p'}(\Omega \times [0, T])$ for some $p' \geq \frac{p}{p-1}$, where p is the constant in Assumption 2.1.

Proposition 2.3. *Let S_t follow the dynamics (2.1), and suppose the objective is (2.3) with power utility function (2.5). Under Assumption 2.1 and 2.2, the value process V_t is given by*

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma} \left[\widetilde{\mathbb{E}} \left(e^{\frac{1-\gamma}{2q\gamma} \int_t^T \lambda^2(Y_s) ds} \middle| \mathcal{G}_t \right) \right]^q. \quad (2.13)$$

The expectation $\tilde{\mathbb{E}}[\cdot]$ is computed with respect to $\tilde{\mathbb{P}}$ introduced in (2.9). The parameter q is given in term of γ and ρ by (2.7). The optimal strategy π^* is

$$\pi_t^* = \left[\frac{\lambda(Y_t)}{\gamma\sigma(Y_t)} + \frac{\rho q \xi_t}{\gamma\sigma(Y_t)} \right] X_t, \quad (2.14)$$

where ξ_t is given in (2.12).

The conditioning with respect to \mathcal{G}_t corresponds to the separation of variable in the Markovian case presented in Section 2.1.

Remark 2.4.

(i) Note that $\gamma = 1$ in (2.5) is the log utility case, which can be treated separately.

(ii) For the degenerate case $\lambda(y) \equiv \lambda_0$, the value process V_t is reduced to

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma} \lambda_0^2 (T-t)}.$$

The quantity $a_t = -\rho \left(\frac{1-\gamma}{\gamma} \right) \lambda_0$ is a constant and direct computation from (2.11) yields $\xi_t = 0$. Consequently, the optimal control π^* becomes

$$\pi_t^* = \frac{\lambda_0}{\gamma\sigma(Y_t)} X_t.$$

In this case, both V_t and π_t^* do not depend on a_t and q as expected.

(iii) In the uncorrelated case $\rho = 0$, the problem is already “linear”, since $q = 1$. The value process V_t and the optimal control π^* are simplified as

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma} \mathbb{E} \left[e^{\frac{1-\gamma}{2\gamma} \int_t^T \lambda^2(Y_s) ds} \middle| \mathcal{G}_t \right], \quad \pi_t^* = \frac{\lambda(Y_t)}{\gamma\sigma(Y_t)} X_t.$$

Proof of Proposition 2.3. The proof follows a verification argument, that is, in order to prove that V_t is indeed the value process and π^* given in (2.14) is optimal, one needs to prove (i) for any control $\pi_t \in \mathcal{A}_t$ satisfying Assumption 2.1, the process (2.13) is a supermartingale, and (ii) V_t is a martingale under the control (2.14) which needs to be admissible.

Let α_t be the proportion of the wealth invested in S_t at time t , namely, $\pi_t = \alpha_t X_t$, then the wealth process (2.2) can be rewritten as:

$$dX_t = X_t [\alpha_t \mu(Y_t) dt + \alpha_t \sigma(Y_t) dW_t]. \quad (2.15)$$

In the following proof, we shall first derive the drift part of dV_t , then obtain α_t^* by maximizing the drift over α , and finally show that the drift part corresponding to α_t^* is zero with the right choice of a_t and q .

Recall the $\tilde{\mathbb{P}}$ -martingale M_t defined in (2.11), and rewrite V_t using M_t as

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma} e^{N_t} M_t^q, \quad (2.16)$$

where $N_t = -\frac{1-\gamma}{2\gamma} \int_0^t \lambda^2(Y_s) ds$. In the following derivation, we use the short notation $\lambda = \lambda(Y_t)$, $\mu =$

$\mu(Y_t), \sigma = \sigma(Y_t)$. By Itô's formula applied to V_t in (2.16), we deduce

$$\begin{aligned}
dV_t &= \left(X_t^{-\gamma} dX_t - \frac{\gamma}{2} X_t^{-\gamma-1} d\langle X \rangle_t \right) e^{N_t} M_t^q + \frac{X_t^{1-\gamma}}{1-\gamma} e^{N_t} M_t^q dN_t + \frac{X_t^{1-\gamma}}{1-\gamma} e^{N_t} q M_t^{q-1} dM_t \\
&\quad + \frac{1}{2} \frac{X_t^{1-\gamma}}{1-\gamma} e^{N_t} q(q-1) M_t^{q-2} d\langle M \rangle_t + d\left\langle \frac{X_t^{1-\gamma}}{1-\gamma} e^{N_t}, M^q \right\rangle_t \\
&= \left(X_t^{-\gamma} X_t \alpha_t \mu - \frac{\gamma}{2} X_t^{-\gamma-1} X_t^2 \alpha_t^2 \sigma^2 \right) e^{N_t} M_t^q dt + \frac{X_t^{1-\gamma}}{1-\gamma} e^{N_t} M_t^q \left(-\frac{1-\gamma}{2\gamma} \lambda^2 \right) dt \\
&\quad + \frac{X_t^{1-\gamma}}{1-\gamma} e^{N_t} q M_t^{q-1} M_t \xi_t a_t dt + \frac{1}{2} \frac{X_t^{1-\gamma}}{1-\gamma} e^{N_t} q(q-1) M_t^{q-2} M_t^2 \xi_t^2 dt + X_t^{-\gamma} e^{N_t} q M_t^{q-1} \rho X_t \alpha_t \sigma M_t \xi_t dt \\
&\quad + \frac{X_t^{1-\gamma}}{1-\gamma} e^{N_t} M_t^q [(1-\gamma) \alpha_t \sigma dW_t + q \xi_t dW_t^Y].
\end{aligned}$$

Under Assumptions 2.1 and 2.2, the last term is a true martingale. This follows from the boundedness of $e^{N_t} M_t^q$ guaranteed by the boundedness of $\lambda(\cdot)$, and square integrability of $X_t^{1-\gamma} \alpha_t \sigma$ and $X_t^{1-\gamma} \xi_t$ implied by:

$$\begin{aligned}
\mathbb{E} \left[\int_0^T (X_t^\pi)^{2-2\gamma} a_t^2 \sigma^2(Y_t) dt \right] &= \mathbb{E} \left[\int_0^T (X_t^\pi)^{-2\gamma} \pi_t^2 \sigma^2(Y_t) dt \right] < \infty, \\
\mathbb{E} \left[\int_0^T (X_t^\pi)^{2-2\gamma} \xi_t^2 dt \right] &\leq \left[\mathbb{E} \int_0^T (X_t^\pi)^{2p(1-\gamma)} dt \right]^{\frac{1}{p}} \left[\mathbb{E} \int_0^T \xi_t^{2p/(p-1)} dt \right]^{\frac{p-1}{p}} < \infty.
\end{aligned}$$

By rewriting $dV_t = X_t^{1-\gamma} e^{N_t} M_t^q D_t(\alpha_t) dt + d\text{Martingale}$, the drift factor $D_t(\alpha_t)$ takes the form:

$$D_t(\alpha_t) := \alpha_t \mu - \frac{\gamma}{2} \alpha_t^2 \sigma^2 - \frac{\lambda^2}{2\gamma} + \frac{q}{1-\gamma} a_t \xi_t + \frac{q(q-1)}{2(1-\gamma)} \xi_t^2 + \rho q \alpha_t \sigma \xi_t.$$

Differentiating $D_t(\alpha_t)$ with respect to α and checking the second order condition, one obtains the maximizer

$$\alpha_t^* = \frac{\mu}{\gamma \sigma^2} + \frac{\rho q \xi_t}{\gamma \sigma} = \frac{\lambda}{\gamma \sigma} + \frac{\rho q \xi_t}{\gamma \sigma}. \quad (2.17)$$

Evaluating the drift factor D_t at α_t^* produces

$$D_t(\alpha_t^*) = q \xi_t \left(\frac{a_t}{1-\gamma} + \frac{\lambda \rho}{\gamma} \right) + \frac{q \xi_t^2}{2} \left[\frac{\rho^2 q}{\gamma} + \frac{q-1}{1-\gamma} \right]. \quad (2.18)$$

Then, the drift factor $D_t(\alpha_t^*)$ vanishes under the choices (2.7) for q and (2.10) for a_t .

In addition, using the relation $\pi_t = \alpha_t X_t$ and equation (2.17) for α_t^* , the wealth process following π_t^* solves the SDE

$$dX_t^{\pi^*} = X_t^{\pi^*} \left[\frac{\lambda^2(Y_t) + \rho q \lambda(Y_t) \xi_t}{\gamma} dt + \frac{\lambda(Y_t) + \rho q \xi_t}{\gamma} dW_t \right],$$

thus stays non-negative, which implies the admissibility of $\pi_t^* = \alpha_t^* X_t$. \square

Remark 2.5. The choice of a_t and q in (2.10) and (2.7) is consistent with the distortion transformation in the Markovian case reviewed in Section 2.1. In fact, other choices only lead to the degenerate case mentioned in Remark 2.4(ii) by the following argument.

From (2.18), one can factorize the expression and obtain

$$D_t(a_t^*) = q \xi_t \left[\frac{a_t}{1-\gamma} + \frac{\lambda \rho}{\gamma} + \xi_t \left(\frac{\rho^2 q}{2\gamma} + \frac{q-1}{2(1-\gamma)} \right) \right],$$

except the choice (2.10) and (2.7), one could consider: (a) $\xi_t = 0$, and (b) $\frac{a_t}{1-\gamma} + \frac{\lambda \rho}{\gamma} + \xi_t \left(\frac{\rho^2 q}{2\gamma} + \frac{q-1}{2(1-\gamma)} \right) = 0$, given $q \neq \frac{\gamma}{\gamma + (1-\gamma)\rho^2}$.

(a) Suppose $\xi_t = 0$, by solving the martingale representation (2.12) for M_t , one has

$$M_t = M_0 = \widetilde{\mathbb{E}} \left[e^{\frac{1-\gamma}{2q\gamma} \int_0^T \lambda^2(Y_s) ds} \right], \forall t \in [0, T],$$

and M_t is deterministic, which is only valid under the situation in Remark 2.4(ii), where $\lambda(y)$ is a constant.

(b) The process ξ_t given in (2.12) does not depend on the choice of a_t , since by varying a_t , one only varies the drift of M_t in (2.12), but not the diffusion part. Therefore the relation $\frac{a_t}{1-\gamma} + \frac{\lambda \rho}{\gamma} + \xi_t \left(\frac{\rho^2 q}{2\gamma} + \frac{q-1}{2(1-\gamma)} \right) = 0$ cannot hold if $q \neq \frac{\gamma}{\gamma + (1-\gamma)\rho^2}$.

Note that with choice (2.7) for q , the term ξ_t^2 is canceled which corresponds to the cancellation of the nonlinear term $(\partial_y \Phi)^2$ in the PDE argument reviewed in Section 2.1.

2.3 Martingale Distortion Transformation with Multiple Assets

Proposition 2.3 can be generalized to the case of multiple risky assets modeled by

$$dS_t^i = \mu^i(Y_t^i) S_t^i dt + \sum_{j=1}^k \sigma_{ij}(Y_t^i) S_t^i dW_t^j, \quad i = 1, 2, \dots, n. \quad (2.19)$$

Here, each S_t^i is driven by its own stochastic factor Y_t^i , but all factors are adapted to the same single Brownian motion W_t^Y with the correlation structure:

$$d\langle W^i, W^j \rangle_t = 0, \quad d\langle W^i, W^Y \rangle_t = \rho dt, \quad \forall i, j = 1, 2, \dots, n.$$

Denote by $\pi = [\pi^1, \pi^2, \dots, \pi^n]^\dagger \in \mathcal{F}_t$ the trading vector such that π_t^i represents the amount of money invested into S_t^i at time t (\dagger denotes the matrix transpose). In this multi-asset setup, under self-financing assumption and $r = 0$, the wealth process X_t satisfies

$$dX_t = \sum_{i=1}^n \pi_t^i \mu^i(Y_t^i) dt + \sum_{i=1}^n \pi_t^i \sum_{j=1}^k \sigma_{ij}(Y_t^i) dW_t^j = \pi_t \cdot \mu(\mathbf{Y}_t) dt + \pi_t \cdot \sigma(\mathbf{Y}_t) dW_t,$$

with vector notations $\mathbf{Y}_t := [Y_t^1, Y_t^2, \dots, Y_t^n]^\dagger$, $\mu(\mathbf{Y}_t) := [\mu^1(Y_t^1), \mu^2(Y_t^2), \dots, \mu^n(Y_t^n)]^\dagger$, $\sigma(\mathbf{Y}_t) := \sigma_{i,j}(Y_t^i)$ as an $n \times k$ matrix, and $W_t := [W_t^1, W_t^2, \dots, W_t^k]^\dagger$.

The assumptions of the multi-assets case are similar to Assumption 2.1 and 2.2, which are summarized as follows.

Assumption 2.6 (Power Utility with Multi-assets). *The admissible trading vectors $\pi \in \mathcal{A}_t$ satisfy*

$$\mathbb{E} \left[\sup_{t \in [0, T]} (X_t^\pi)^{2p(1-\gamma)} \right] < +\infty, \text{ for some } p > 1, \quad \text{and} \quad \mathbb{E} \left[\int_0^T (X_t^\pi)^{-2\gamma} \|\pi_t \cdot \sigma(Y_t)\|^2 dt \right] < \infty.$$

Assumption 2.7. *The SDEs (2.19) for S_t^i have unique strong solutions for all $i = 1, 2, \dots, n$. Assume that the matrix $\sigma(\cdot)$ is of rank n , $\Sigma(\cdot) := \sigma(\cdot)\sigma(\cdot)^\dagger$ is invertible and positive definite. The function $\Lambda(\cdot) := \mu(\cdot)^\dagger \Sigma(\cdot)^{-1} \mu(\cdot)$ is bounded and $C^2(\mathbb{R})$, and its derivatives $\Lambda'(\cdot)$ and $\Lambda''(\cdot)$ are at most polynomially growing.*

Moreover, define the $\widetilde{\mathbb{P}}$ -martingale M_t by

$$M_t = \widetilde{\mathbb{E}} \left(e^{\frac{1-\gamma}{2q\gamma} \int_0^T \mu(\mathbf{Y}_s)^\dagger \Sigma(\mathbf{Y}_s)^{-1} \mu(\mathbf{Y}_s) ds} \middle| \mathcal{G}_t \right),$$

where $\widetilde{\mathbb{E}}$ is the expectation under the probability measure (2.9) $\widetilde{\mathbb{P}}$ with

$$a_t = -\rho \left(\frac{1-\gamma}{\gamma} \right) \mathbf{1}_k^\dagger \sigma(\mathbf{Y}_t)^\dagger \Sigma^{-1}(\mathbf{Y}_t) \mu(\mathbf{Y}_t). \quad (2.20)$$

Then, one has

$$dM_t = M_t \xi_t d\widetilde{W}_t^Y, \quad (2.21)$$

and we require that $\xi \in L^{2p'}(\Omega \times [0, T])$ for some constant $p' > \frac{p}{p-1}$, where p is the constant in Assumption 2.6.

Proposition 2.8. *Under Assumption 2.6 and 2.7, the portfolio value V_t can be expressed as*

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma} \left[\tilde{\mathbb{E}} \left(e^{\frac{1-\gamma}{2q\gamma} \int_t^T \mu(\mathbf{Y}_s)^\dagger \Sigma(\mathbf{Y}_s)^{-1} \mu(\mathbf{Y}_s) ds} \middle| \mathcal{G}_t \right) \right]^q,$$

where $\tilde{\mathbb{E}}$ is calculated under $\tilde{\mathbb{P}}$ with a_t in (2.20), the constant q is chosen to be:

$$q = \frac{\gamma}{\gamma + (1-\gamma)\rho^2 \mathbf{1}_k^\dagger \sigma(\mathbf{Y}_t)^\dagger \Sigma^{-1}(\mathbf{Y}_t) \sigma(\mathbf{Y}_t) \mathbf{1}_k},$$

and $\mathbf{1}_k$ is a k -vector of ones. The optimal control π^* is given by

$$\pi_t^* = \left[\frac{\Sigma(\mathbf{Y}_t)^{-1} \mu(\mathbf{Y}_t)}{\gamma} + \frac{\rho q \xi_t \Sigma(\mathbf{Y}_t)^{-1} \sigma(\mathbf{Y}_t) \mathbf{1}_k}{\gamma} \right] X_t,$$

with ξ given in (2.21).

Proof. The proof is a straightforward generalization of the one given in the single asset case, thus we omit the details here. Note that the case $n = k = 1$ is reduced to Proposition 2.3. \square

3 Application to Fractional Stochastic Environment

In this section, we first briefly review the fractional Brownian motion (fBm) and fractional Ornstein-Uhlenbeck (fOU) processes, and then introduce the slowly varying fOU process. Under such a model, we will derive an approximation of portfolio value V_t based on results in Proposition 2.3. More importantly, note that the optimal trading strategy π^* given by (2.14) is not explicit due to the presence of ξ_t given by the martingale representation theorem, and we will obtain an explicit approximation to this optimal strategy.

3.1 Fractional Brownian Motion and Fractional Ornstein-Uhlenbeck Processes

A fractional Brownian motion is a continuous Gaussian process $(W_t^{(H)})$ with zero mean and the covariance structure:

$$\mathbb{E} \left[W_t^{(H)} W_s^{(H)} \right] = \frac{\sigma_H^2}{2} \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right), \quad (3.1)$$

where σ_H is a positive constant and $H \in (0, 1)$ is called Hurst index. According to Mandelbrot and Van Ness [1968], $W_t^{(H)}$ has the following moving-average stochastic integral representation:

$$W_t^{(H)} = \frac{1}{\Gamma(H + \frac{1}{2})} \int_{\mathbb{R}} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dW_s, \quad (3.2)$$

where $(W_t)_{t \in \mathbb{R}^+}$ is the usual Brownian motion and $(W_t)_{t \in \mathbb{R}^-} := (B_{-t})_{t \in \mathbb{R}^-}$ is another Brownian motion independent of $(W_t)_{t \in \mathbb{R}^+}$. Therefore, σ_H^2 is calculated as:

$$\sigma_H^2 = \frac{1}{\Gamma^2(H + \frac{1}{2})} \left[\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right)^2 ds + \frac{1}{2H} \right] = \frac{1}{\Gamma(2H+1) \sin(\pi H)}. \quad (3.3)$$

Now we consider the Langevin equation with fractional Brownian motion

$$dZ_t^H = -aZ_t^H dt + dW_t^{(H)}, \quad (3.4)$$

with the initial condition $Z_0^H = \eta$. In Cheridito et al. [2003], it is proved that

$$Z_t^{H,\eta} := e^{-at} \left(\eta + \int_0^t e^{au} dW_u^{(H)} \right)$$

is the unique almost surely continuous process that solves equation (3.4), where $\int_0^t e^{au} dW_u^{(H)}$ exists as a path-wise Riemann-Stieltjes integral (by integration by parts) and is almost surely continuous in t . Particularly, for $t \in \mathbb{R}^+$,

$$Z_t^H := \int_{-\infty}^t e^{-a(t-s)} dW_s^{(H)} = W_t^{(H)} - a \int_{-\infty}^t e^{-a(t-s)} W_s^{(H)} ds, \quad (3.5)$$

is a stationary solution with initial condition $\eta = Z_0^H$, and every other stationary solution has the same distribution as Z_t^H . In the sequel, we shall only consider this stationary solution and call it the *stationary fractional Ornstein-Uhlenbeck process*.

It has zero mean and (co)variance structure:

$$\sigma_{ou}^2 = \frac{1}{2} a^{-2H} \Gamma(2H+1) \sigma_H^2, \quad (3.6)$$

$$\mathbb{E} [Z_t^H Z_{t+s}^H] = \sigma_{ou}^2 \frac{2 \sin(\pi H)}{\pi} \int_0^\infty \cos(asx) \frac{x^{1-2H}}{1+x^2} dx := \sigma_{ou}^2 \mathcal{C}_Z(s). \quad (3.7)$$

Using the moving-average representation (3.2) for $W_t^{(H)}$, the stationary solution (3.5) can be expressed as:

$$Z_t^H = \int_{-\infty}^t \mathcal{K}(t-s) dW_s^Z, \quad (3.8)$$

where $(W_t^Z)_{t \in \mathbb{R}}$ is a standard BM on \mathbb{R} as described in (3.2), with the superscript Z indicating that it drives the process Z_t^H . The kernel \mathcal{K} is defined by

$$\mathcal{K}(t) = \frac{1}{\Gamma(H + \frac{1}{2})} \left[t^{H-\frac{1}{2}} - a \int_0^t (t-s)^{H-\frac{1}{2}} e^{-as} ds \right]. \quad (3.9)$$

We refer to [Garnier and Solna, 2015, Section 2.2] for asymptotic properties of $\mathcal{K}(t)$ when $t \ll 1$ and $t \gg 1$, for short-range correlation properties when $H \in (0, \frac{1}{2})$, and for long-range correlation properties when $H \in (\frac{1}{2}, 1)$. In what follows, we will be mainly interested in the case $H < \frac{1}{2}$ as explained in the introduction, but our asymptotic results are also valid for $H > \frac{1}{2}$. As noted in [Garnier and Solna, 2015, Appendix B], a more general class of Gaussian volatility factors can be considered. But for the sake of simplicity and length, we restrict ourselves to the case of fOU process.

3.2 The Slowly Varying fOU Process

As explained in the introduction, we consider the slowly varying fractional factor denoted by $Z_t^{\delta, H}$. In the regime of small δ , $Z_t^{\delta, H}$ is defined as a rescaled stationary fOU process,

$$Z_t^{\delta, H} = \delta^H \int_{-\infty}^t e^{-\delta a(t-s)} dW_s^{(H)} = \int_{-\infty}^t \mathcal{K}^\delta(t-s) dW_s^Z, \quad \mathcal{K}^\delta(t) = \sqrt{\delta} \mathcal{K}(\delta t), \quad (3.10)$$

where $W_t^{(H)}$ is a fBm driven by the Brownian motion W_t^Z via (3.2), and $\mathcal{K}(t)$ is given in (3.9). According to Section 3.1, $Z_t^{\delta, H}$ is a stationary solution to the SDE

$$dZ_t^{\delta, H} = -\delta a Z_t^{\delta, H} dt + \delta^H dW_t^{(H)}. \quad (3.11)$$

It is a zero-mean, stationary Gaussian process with variance σ_{ou}^2 and covariance

$$\mathbb{E} [Z_t^{\delta, H} Z_{t+s}^{\delta, H}] = \sigma_{ou}^2 \mathcal{C}_Z(\delta s) = \sigma_{ou}^2 \frac{2 \sin(\pi H)}{\pi} \int_0^\infty \cos(\delta a s x) \frac{x^{1-2H}}{1+x^2} dx. \quad (3.12)$$

The covariance function depends on δs only, which indicates that $1/\delta$ is the natural scale of $Z_t^{\delta, H}$ as desired. More properties and estimates regarding $Z_t^{\delta, H}$ are stated in Lemma A.1.

As δ goes to zero, by dominated convergence theorem, the covariance becomes

$$\lim_{\delta \rightarrow 0} \mathbb{E} [Z_t^{\delta, H} Z_{t+s}^{\delta, H}] = \sigma_{ou}^2 \frac{2 \sin(\pi H)}{\pi} \int_0^\infty \frac{x^{1-2H}}{1+x^2} dx = \sigma_{ou}^2 \frac{2 \sin(\pi H)}{\pi} \frac{\pi}{2} \csc(\pi H) = \sigma_{ou}^2, \quad (3.13)$$

and the process $Z_t^{\delta, H}$ converges in distribution to $\left(Z_0^{\delta, H}\right)_{t \in \mathbb{R}} \stackrel{\mathcal{D}}{=} (\sigma_{ou} Z)_{t \in \mathbb{R}}$, where Z is a standard normal random variable.

3.3 First order Approximation to the Value Process

In this section, we study the problem discussed in Section 2 with $Y_t = Z_t^{\delta, H}$ and $W_t^Y = W_t^Z$. To be precise, the underlying asset S_t is driven by the slowly varying fractional stochastic factor $Z_t^{\delta, H}$ defined in (3.10),

$$dS_t = \mu(Z_t^{\delta, H}) S_t dt + \sigma(Z_t^{\delta, H}) S_t dW_t.$$

Still, we denote by X_t^π the wealth process, and it follows

$$dX_t^\pi = \pi_t \mu(Z_t^{\delta, H}) dt + \pi_t \sigma(Z_t^{\delta, H}) dW_t.$$

The value process is denoted by V_t^δ to indicate its dependence of δ introduced by the slowly varying process $Z_t^{\delta, H}$:

$$V_t^\delta := \sup_{\pi \in \mathcal{A}_t} \mathbb{E} [U(X_T^\pi) | \mathcal{F}_t].$$

Note that, by definition, the process $Z_t^{\delta, H}$ is neither Markovian nor a semimartingale when $H \neq \frac{1}{2}$, therefore the HJB equation is not available. However, it is adapted to \mathcal{G}_t . In order to use Proposition 2.3, we need to check that $Z_t^{\delta, H}$ satisfies Assumption 2.2(ii).

Lemma 3.1. *The slowly varying fractional factor $Z_t^{\delta, H}$ defined in (3.10) satisfies the Assumption 2.2(ii).*

Proof. To obtain the process $(\xi_t)_{t \in [0, T]}$ in (2.12), we shall use Malliavin calculus. By the Clark-Ocone Formula under change of measure (see Di Nunno et al. [2009]), we obtain

$$M_t \xi_t = \tilde{\mathbb{E}} \left[\mathcal{D}_t M_T - M_T \int_t^T \mathcal{D}_t a_s d\tilde{W}_s^Z \middle| \mathcal{G}_t \right] := \tilde{\mathbb{E}} [M_T F_{[t, T]} | \mathcal{G}_t],$$

where \mathcal{D}_t denotes the Malliavian derivative, and $F_{[t, T]}$ is

$$\begin{aligned} F_{[t, T]} &= \frac{1-\gamma}{q\gamma} \int_t^T \lambda(Z_s^{\delta, H}) \lambda'(Z_s^{\delta, H}) \mathcal{K}^\delta(s-t) ds + \rho \frac{1-\gamma}{\gamma} \int_t^T \lambda'(Z_s^{\delta, H}) \mathcal{K}^\delta(s-t) d\tilde{W}_s^Z \\ &= \frac{1-\gamma}{\gamma} \int_t^T \lambda(Z_s^{\delta, H}) \lambda'(Z_s^{\delta, H}) \mathcal{K}^\delta(s-t) ds + \rho \frac{1-\gamma}{\gamma} \int_t^T \lambda'(Z_s^{\delta, H}) \mathcal{K}^\delta(s-t) dW_s^Z. \end{aligned}$$

Thus, $\xi_t = \tilde{\mathbb{E}} [M_T M_t^{-1} F_{[t, T]} | \mathcal{G}_t]$. Then Assumption 2.2(i) and Lemma A.1(i) imply that $\xi_t \in L^k(\Omega \times [0, T])$, for any $k \geq 2$:

$$\begin{aligned} \mathbb{E} \left[\int_0^T |\xi_t|^k dt \right] &= \int_0^T \mathbb{E} \left[\left| \tilde{\mathbb{E}} [M_T M_t^{-1} F_{[t, T]} | \mathcal{G}_t] \right|^k \right] dt \leq C \int_0^T \mathbb{E} \left[|F_{[t, T]}|^k e^{-\int_t^T a_s dW_s - \frac{1}{2} \int_t^T a_s^2 ds} \right] dt \\ &\leq C' \int_0^T \mathbb{E} [F_{[t, T]}^{2k}] dt < \infty. \end{aligned}$$

In the derivation, we have used the facts that M_t is a bounded positive $\tilde{\mathbb{P}}$ -martingale and the $(2k)^{th}$ moments of $F_{[t, T]}$ exist for any k . The second fact follows from the polynomial growth property of $\lambda(\cdot)$, existence of moments of $Z_t^{\delta, H}$ and uniformly integrability of $\mathcal{K}^\delta(t)$ on $[0, T]$. \square

Theorem 3.2. *Under Assumption 2.1 and 2.2, for fixed $t \in [0, T]$, $X_t = x$ and the observed value $Z_0^{\delta, H}$, V_t^δ takes the form*

$$V_t^\delta = Q_t^\delta(X_t, Z_0^{\delta, H}) + \mathcal{O}(\delta^{2H}), \quad (3.14)$$

where

$$Q_t^\delta(x, z) = \frac{x^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma} \lambda^2(z)(T-t)} \left[1 + \frac{1-\gamma}{\gamma} \lambda(z) \lambda'(z) \left(\phi_t^\delta + \delta^H \rho \lambda(z) \left(\frac{1-\gamma}{\gamma} \right) \frac{(T-t)^{H+\frac{3}{2}}}{\Gamma(H+\frac{5}{2})} \right) \right]. \quad (3.15)$$

Here ϕ_t^δ is defined by

$$\phi_t^\delta = \mathbb{E} \left[\int_t^T (Z_s^{\delta, H} - Z_0^{\delta, H}) \, ds \middle| \mathcal{F}_t \right] = \mathbb{E} \left[\int_t^T (Z_s^{\delta, H} - Z_0^{\delta, H}) \, ds \middle| \mathcal{G}_t \right], \quad (3.16)$$

and ϕ_t^δ is of order δ^H as proved in Lemma A.1 in the sense that its variance is of order δ^{2H} . Note that $\mathcal{O}(\delta^{2H})$ denotes a \mathcal{F}_t -adapted random variable and it is of order δ^{2H} in L^2 .

Proof. A straightforward application of Proposition 2.3 with $Y_t = Z_t^{\delta, H}$ and $W_t^Y = W_t^Z$ gives the following representation of the value process V_t^δ

$$V_t^\delta = \frac{X_t^{1-\gamma}}{1-\gamma} \left[\widetilde{\mathbb{E}} \left(e^{\frac{1-\gamma}{2q\gamma} \int_t^T \lambda^2(Z_s^{\delta, H}) \, ds} \middle| \mathcal{G}_t \right) \right]^q.$$

We start by expanding $\Psi_t^\delta := \widetilde{\mathbb{E}} \left[e^{\frac{1-\gamma}{2q\gamma} \int_t^T \lambda^2(Z_s^{\delta, H}) \, ds} \middle| \mathcal{G}_t \right]$, and then apply Taylor formula to the function x^q .

The formula for the conditional expectation under an absolute continuous change of measure, together with the value of a_t given by (2.10) and Taylor expansion in z at the point $Z_0^{\delta, H}$ yields,

$$\begin{aligned} \Psi_t^\delta &= \mathbb{E} \left[e^{\frac{1-\gamma}{2q\gamma} \int_t^T \lambda^2(Z_s^{\delta, H}) \, ds} e^{-\int_t^T a_s \, dW_s^Z - \frac{1}{2} \int_t^T a_s^2 \, ds} \middle| \mathcal{G}_t \right] \\ &= \mathbb{E} \left[e^{\frac{1-\gamma}{2q\gamma} \int_t^T \lambda^2(Z_s^{\delta, H}) \, ds} e^{\int_t^T \rho \left(\frac{1-\gamma}{\gamma} \right) \lambda(Z_s^{\delta, H}) \, dW_s^Z - \frac{1}{2} \int_t^T \rho^2 \left(\frac{1-\gamma}{\gamma} \right)^2 \lambda^2(Z_s^{\delta, H}) \, ds} \middle| \mathcal{G}_t \right] \\ &= e^{\frac{1-\gamma}{2q\gamma} \lambda^2(Z_0^{\delta, H})(T-t)} \mathbb{E} \left[e^{\int_t^T \rho \left(\frac{1-\gamma}{\gamma} \right) \lambda(Z_0^{\delta, H}) \, dW_s^Z - \frac{1}{2} \int_t^T \rho^2 \left(\frac{1-\gamma}{\gamma} \right)^2 \lambda^2(Z_0^{\delta, H}) \, ds + A_{[t, T]} + B_{[t, T]}} \middle| \mathcal{G}_t \right], \end{aligned}$$

where $A_{[t, T]}$ and $B_{[t, T]}$ are given by

$$\begin{aligned} A_{[t, T]} &= \frac{1-\gamma}{q\gamma} \lambda(Z_0^{\delta, H}) \lambda'(Z_0^{\delta, H}) \int_t^T (Z_s^{\delta, H} - Z_0^{\delta, H}) \, ds + \rho \left(\frac{1-\gamma}{\gamma} \right) \lambda'(Z_0^{\delta, H}) \int_t^T (Z_s^{\delta, H} - Z_0^{\delta, H}) \, dW_s^Z \\ &\quad - \rho^2 \left(\frac{1-\gamma}{\gamma} \right)^2 \lambda(Z_0^{\delta, H}) \lambda'(Z_0^{\delta, H}) \int_t^T (Z_s^{\delta, H} - Z_0^{\delta, H}) \, ds, \\ B_{[t, T]} &= \frac{1-\gamma}{q\gamma} \int_t^T (\lambda \lambda'' + \lambda'^2) (\chi_s) (Z_s^{\delta, H} - Z_0^{\delta, H})^2 \, ds + \rho \left(\frac{1-\gamma}{\gamma} \right) \int_t^T \lambda''(\eta_s) (Z_s^{\delta, H} - Z_0^{\delta, H})^2 \, dW_s^Z \\ &\quad - \rho^2 \left(\frac{1-\gamma}{\gamma} \right)^2 \int_t^T (\lambda \lambda'' + \lambda'^2) (\chi_s) (Z_s^{\delta, H} - Z_0^{\delta, H})^2 \, ds, \end{aligned}$$

with χ_s and η_s being the Lagrange remainders: $\chi_s, \eta_s \in [Z_0^{\delta, H} \vee Z_s^{\delta, H}, Z_0^{\delta, H} \wedge Z_s^{\delta, H}]$.

Since $\lambda(\cdot)$ is bounded, one can expand $e^{A_{[t, T]} + B_{[t, T]}}$ and deduce

$$\begin{aligned} \Psi_t^\delta &= e^{\frac{1-\gamma}{2q\gamma} \lambda^2(Z_0^{\delta, H})(T-t)} \mathbb{E} \left[e^{\int_t^T \rho \left(\frac{1-\gamma}{\gamma} \right) \lambda(Z_0^{\delta, H}) \, dW_s^Z - \frac{1}{2} \int_t^T \rho^2 \left(\frac{1-\gamma}{\gamma} \right)^2 \lambda^2(Z_0^{\delta, H}) \, ds} (1 + A_{[t, T]} + R_{[t, T]}) \middle| \mathcal{G}_t \right] \\ &= e^{\frac{1-\gamma}{2q\gamma} \lambda^2(Z_0^{\delta, H})(T-t)} \mathbb{E} \left[e^{\int_t^T \rho \left(\frac{1-\gamma}{\gamma} \right) \lambda(Z_0^{\delta, H}) \, dW_s^Z - \frac{1}{2} \int_t^T \rho^2 \left(\frac{1-\gamma}{\gamma} \right)^2 \lambda^2(Z_0^{\delta, H}) \, ds} (1 + A_{[t, T]}) \middle| \mathcal{G}_t \right] + \mathcal{O}(\delta^{2H}), \end{aligned}$$

where $R_{[t, T]}$ is given by

$$R_{[t, T]} = B_{[t, T]} + \sum_{k=2}^{\infty} \frac{1}{k!} (A_{[t, T]} + B_{[t, T]})^k,$$

and $O(\delta^{2H})$ is a random variable of order δ^{2H} in L^2 sense as mentioned before.

We introduce a new probability measure $\widehat{\mathbb{P}}$, such that under $\widehat{\mathbb{P}}$, $\widehat{W}_t^Z = W_t^Z - \rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Z_0^{\delta,H})t$ is a standard Brownian motion. Then Ψ_t^δ can be rewritten as

$$\begin{aligned}\Psi_t^\delta &= e^{\frac{1-\gamma}{2q\gamma}\lambda^2(Z_0^{\delta,H})(T-t)}\widehat{\mathbb{E}}\left[(1+A_{[t,T]})\middle|\mathcal{G}_t\right] + \mathcal{O}(\delta^{2H}) \\ &= e^{\frac{1-\gamma}{2q\gamma}\lambda^2(Z_0^{\delta,H})(T-t)}\widehat{\mathbb{E}}\left[1 + \frac{(1-\gamma)}{q\gamma}\lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H})\int_t^T (Z_s^{\delta,H} - Z_0^{\delta,H})\,ds\middle|\mathcal{G}_t\right] \\ &\quad + e^{\frac{1-\gamma}{2q\gamma}\lambda^2(Z_0^{\delta,H})(T-t)}\widehat{\mathbb{E}}\left[\rho\left(\frac{1-\gamma}{\gamma}\right)\lambda'(Z_0^{\delta,H})\int_t^T (Z_s^{\delta,H} - Z_0^{\delta,H})\,dW_s^Z\middle|\mathcal{G}_t\right] \\ &\quad - e^{\frac{1-\gamma}{2q\gamma}\lambda^2(Z_0^{\delta,H})(T-t)}\widehat{\mathbb{E}}\left[\rho^2\left(\frac{1-\gamma}{\gamma}\right)^2\lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H})\int_t^T (Z_s^{\delta,H} - Z_0^{\delta,H})\,ds\middle|\mathcal{G}_t\right] + \mathcal{O}(\delta^{2H}),\end{aligned}$$

and the second term is canceled with the third one, since

$$\begin{aligned}&\widehat{\mathbb{E}}\left[\rho\left(\frac{1-\gamma}{\gamma}\right)\lambda'(Z_0^{\delta,H})\int_t^T (Z_s^{\delta,H} - Z_0^{\delta,H})\,dW_s^Z\middle|\mathcal{G}_t\right] \\ &= \widehat{\mathbb{E}}\left[\rho\left(\frac{1-\gamma}{\gamma}\right)\lambda'(Z_0^{\delta,H})\int_t^T (Z_s^{\delta,H} - Z_0^{\delta,H})\,d\widehat{W}_s^Z\middle|\mathcal{G}_t\right] \\ &\quad + \widehat{\mathbb{E}}\left[\rho\left(\frac{1-\gamma}{\gamma}\right)\lambda'(Z_0^{\delta,H})\int_t^T (Z_s^{\delta,H} - Z_0^{\delta,H})\rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Z_0^{\delta,H})\,ds\middle|\mathcal{G}_t\right] \\ &= \widehat{\mathbb{E}}\left[\rho^2\left(\frac{1-\gamma}{\gamma}\right)^2\lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H})\int_t^T (Z_s^{\delta,H} - Z_0^{\delta,H})\,ds\middle|\mathcal{G}_t\right].\end{aligned}$$

Thus, the term Ψ_t^δ is simplified to

$$\Psi_t^\delta = e^{\frac{1-\gamma}{2q\gamma}\lambda^2(Z_0^{\delta,H})(T-t)}\left(1 + \frac{(1-\gamma)}{q\gamma}\lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H})\Phi_t^\delta\right) + \mathcal{O}(\delta^{2H}), \quad (3.17)$$

with

$$\Phi_t^\delta = \widehat{\mathbb{E}}\left[\int_t^T (Z_s^{\delta,H} - Z_0^{\delta,H})\,ds\middle|\mathcal{G}_t\right] = \widehat{\mathbb{E}}\left[\int_t^T Z_s^{\delta,H}\,ds\middle|\mathcal{G}_t\right] - Z_0^{\delta,H}(T-t).$$

To further simplify Φ_t^δ , we use the moving average representation (3.10) for $Z_s^{\delta,H}$ and deduce

$$\begin{aligned}\Phi_t^\delta &= \widehat{\mathbb{E}}\left[\int_t^T Z_s^{\delta,H}\,ds\middle|\mathcal{G}_t\right] - Z_0^{\delta,H}(T-t) = \widehat{\mathbb{E}}\left[\int_t^T \int_{-\infty}^s \mathcal{K}^\delta(s-u)\,dW_u^Z\,ds\middle|\mathcal{G}_t\right] - Z_0^{\delta,H}(T-t) \\ &= \widehat{\mathbb{E}}\left[\int_{-\infty}^t \int_t^T \mathcal{K}^\delta(s-u)\,ds\,dW_u^Z\middle|\mathcal{G}_t\right] + \widehat{\mathbb{E}}\left[\int_t^T \int_u^T \mathcal{K}^\delta(s-u)\,ds\,dW_u^Z\middle|\mathcal{G}_t\right] - Z_0^{\delta,H}(T-t) \\ &= \int_{-\infty}^t \int_t^T \mathcal{K}^\delta(s-u)\,ds\,dW_u^Z - Z_0^{\delta,H}(T-t) + \widehat{\mathbb{E}}\left[\int_t^T \int_u^T \mathcal{K}^\delta(s-u)\,ds\,dW_u^Z\middle|\mathcal{G}_t\right] \\ &= \phi_t^\delta + \widehat{\mathbb{E}}\left[\int_t^T \int_u^T \mathcal{K}^\delta(s-u)\,ds\,d\widehat{W}_u^Z\middle|\mathcal{G}_t\right] + \rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Z_0^{\delta,H})\int_t^T \int_u^T \mathcal{K}^\delta(s-u)\,ds\,du \\ &= \phi_t^\delta + \rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Z_0^{\delta,H})\frac{\delta^H(T-t)^{H+3/2}}{\Gamma(H+\frac{5}{2})} + \mathcal{O}(\delta^{H+1}).\end{aligned} \quad (3.18)$$

In the derivation, we have changed the order of ds and dW_u^Z , and use the relation $\widehat{W}_t^Z = W_t^Z - \frac{1-\gamma}{\gamma}\lambda(Z_0^{\delta,H})\rho t$. Now combining (3.17) and (3.18), we obtain

$$\begin{aligned} V_t^\delta &= \frac{X_t^{1-\gamma}}{1-\gamma} (\Psi_t^\delta)^q \\ &= \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda^2(Z_0^{\delta,H})(T-t)} \left\{ 1 + \frac{1-\gamma}{\gamma}\lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H})\Phi_t^\delta \right\} + O(\delta^{2H}) \\ &= \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda^2(Z_0^{\delta,H})(T-t)} \left\{ 1 + \frac{1-\gamma}{\gamma}\lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H}) \left(\phi_t^\delta + \delta^H \rho \lambda(Z_0^{\delta,H}) \left(\frac{1-\gamma}{\gamma} \right) \frac{(T-t)^{H+\frac{3}{2}}}{\Gamma(H+\frac{5}{2})} \right) \right\} \\ &\quad + O(\delta^{2H}). \end{aligned}$$

Observe that there are two corrections to the leading term: a random component ϕ_t^δ , and a deterministic function of $(t, X_t, Z_0^{\delta,H})$, both being of order δ^H . \square

Remark 3.3 (Discussion of the assumptions on $\lambda(\cdot)$). *In order to expand Ψ_t^δ , we need a uniform bound (in δ) of $\mathbb{E} \left[e^{\frac{1-\gamma}{2q\gamma} \int_t^T \lambda^2(Z_s^{\delta,H}) ds} \right]$. Notice that if $\gamma > 1$, this is automatically satisfied, since the exponential function is bounded by 1. For $0 < \gamma < 1$, it is also satisfied under the assumption $\lambda(\cdot)$ bounded as stated in Assumption 2.2(i). Moreover, the assumption can be relaxed to have uniform bounds for exponential moments of the function $\lambda^2(\cdot)$.*

3.4 Optimal Strategy

We now turn to the expansion to the optimal portfolio given in (2.14)

$$\pi_t^* = \left[\frac{\lambda(Z_t^{\delta,H})}{\gamma\sigma(Z_t^{\delta,H})} + \frac{\rho q \xi_t}{\gamma\sigma(Z_t^{\delta,H})} \right] X_t,$$

where the process ξ_t given by the representation theorem (2.12) is usually not known explicitly. In this section, we approximate ξ_t using the results derived in Theorem 3.2, and we obtain the following asymptotic result for π_t^* .

Theorem 3.4. *Under Assumption 2.1 and 2.2, the optimal strategy π_t^* is approximated by*

$$\begin{aligned} \pi_t^* &= \left[\frac{\lambda(Z_t^{\delta,H})}{\gamma\sigma(Z_t^{\delta,H})} + \delta^H \frac{\rho(1-\gamma)}{\gamma^2\sigma(Z_t^{\delta,H})} \frac{(T-t)^{H+1/2}}{\Gamma(H+\frac{3}{2})} \lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H}) \right] X_t + \mathcal{O}(\delta^{2H}) \\ &:= \pi_t^{(0)} + \delta^H \pi_t^{(1)} + \mathcal{O}(\delta^{2H}). \end{aligned} \tag{3.19}$$

Remark 3.5.

(i) For the case $H = \frac{1}{2}$, $Z_t^{\delta,H}$ becomes the Markovian OU process, and (3.19) coincides with the approximation of feedback form derived in [Fouque et al., 2016, Section 3.2.2 and 6.3.2].

(ii) In the approximation (3.19) to π_t^* , the leading order strategy $\pi_t^{(0)}$ follows the process $Z_t^{\delta,H}$, the first order correction $\pi_t^{(1)}$ is partially frozen at $Z_0^{\delta,H}$, and the random correction ϕ_t^δ appearing in V_t disappears here. This makes the approximated strategy $\pi_t^{(0)} + \delta^H \pi_t^{(1)}$ easier to implement.

Moreover, under additional smoothness assumption on $\sigma(\cdot)$, typically $\sigma(\cdot)$ is C^1 and $(1/\sigma(\cdot))'$ is bounded, then the correction term $\pi_t^{(1)}$ can be fully frozen at $Z_0^{\delta,H}$ without changing the order of accuracy, namely,

$$\pi_t^{(1)} = \frac{\rho(1-\gamma)}{\gamma^2\sigma(Z_0^{\delta,H})} \frac{(T-t)^{H+1/2}}{\Gamma(H+\frac{3}{2})} \lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H}) X_t + \mathcal{O}(\delta^H).$$

(iii) Denote by $X_t^{\pi^{(0)}}$ the wealth process following the zeroth order strategy $\pi_t^{(0)} = \frac{\lambda(Z_t^{\delta,H})}{\gamma\sigma(Z_t^{\delta,H})}X_t$

$$dX_t^{\pi^{(0)}} = \mu(Z_t^{\delta,H})\pi_t^{(0)} dt + \sigma(Z_t^{\delta,H})\pi_t^{(0)} dW_t,$$

and $V_t^{\pi^{(0)},\delta}$ the corresponding value process

$$V_t^{\pi^{(0)},\delta} := \mathbb{E} \left[U \left(X_T^{\pi^{(0)}} \right) \middle| \mathcal{F}_t \right].$$

In Section 4.3 Proposition 4.5, we derive the expansion to $V_t^{\pi^{(0)},\delta}$ for general utility function. When applied to the case of power utility (2.5), one can deduce that $V_t^{\pi^{(0)},\delta} - Q_t^\delta$ is of order δ^{2H} with Q_t^δ given in (3.15). Therefore, by Theorem 3.2, $V_t^{\pi^{(0)},\delta} - V_t^\delta$ is of order δ^{2H} , and we conclude that $\pi_t^{(0)} = \frac{\lambda(Z_t^{\delta,H})}{\gamma\sigma(Z_t^{\delta,H})}$ generates the approximated value process given by (3.14), and is asymptotically optimal within all admissible strategy \mathcal{A}_t up to order δ^H .

Proof. It suffices to derive the expansion of ξ_t determined by (2.12). In the previous section, we have obtained a rigorous expansion for $\Psi_t^\delta := \widetilde{\mathbb{E}} \left[e^{\frac{1-\gamma}{2q\gamma} \int_t^T \lambda^2(Z_s^{\delta,H}) ds} \middle| \mathcal{G}_t \right]$; see (3.17) and (3.18). Rewrite M_t defined in (2.11) using Ψ_t^δ as

$$M_t = e^{I_t} \Psi_t^\delta,$$

where $I_t = \frac{1-\gamma}{2q\gamma} \int_0^t \lambda^2(Z_s^{\delta,H}) ds$. Applying Itô's formula to M_t yields,

$$\begin{aligned} dM_t &= e^{I_t} \Psi_t^\delta dI_t + e^{I_t} d\Psi_t^\delta \\ &= \frac{1-\gamma}{2q\gamma} \lambda^2(Z_t^{\delta,H}) M_t dt + e^{I_t} \left(-\frac{1-\gamma}{2q\gamma} \lambda^2(Z_0^{\delta,H}) \Psi_t^\delta dt + e^{\frac{1-\gamma}{2q\gamma} \lambda^2(Z_0^{\delta,H})(T-t)} \frac{1-\gamma}{q\gamma} \lambda(Z_0^{\delta,H}) \lambda'(Z_0^{\delta,H}) d\Phi_t^\delta \right) \\ &\quad + \mathcal{O}(\delta^{2H}) \\ &= \delta^H e^{I_t} e^{\frac{1-\gamma}{2q\gamma} \lambda^2(Z_0^{\delta,H})(T-t)} \frac{1-\gamma}{q\gamma} \lambda(Z_0^{\delta,H}) \lambda'(Z_0^{\delta,H}) \theta_{t,T} d\widetilde{W}_t^Z + \mathcal{O}(\delta^{2H}). \end{aligned}$$

Here in the derivation, we have successively used the relation (3.17) and (3.18), $d\psi_t^\delta = d\phi_t^\delta + (Z_t^{\delta,H} - Z_0^{\delta,H}) dt$, where ψ_t^δ is given by

$$\psi_t^\delta = \mathbb{E} \left[\int_0^T Z_s^{\delta,H} - Z_0^{\delta,H} ds \middle| \mathcal{F}_t \right] \quad (3.20)$$

and $d\psi_t^\delta = \delta^H \theta_{t,T} dW_t^Z + \delta^{H+1} \widetilde{\theta}_{t,T} dW_t^Z$ with $\theta_{t,T}$ and $\widetilde{\theta}_{t,T}$ specified in Lemma A.1.

Noticing that from (3.17), one can deduce

$$\Psi_t^\delta = e^{\frac{1-\gamma}{2q\gamma} \lambda^2(Z_0^{\delta,H})(T-t)} + \mathcal{O}(\delta^H),$$

then dM_t becomes

$$\begin{aligned} dM_t &= \delta^H e^{I_t} \Psi_t^\delta \frac{1-\gamma}{q\gamma} \lambda(Z_0^{\delta,H}) \lambda'(Z_0^{\delta,H}) \theta_{t,T} d\widetilde{W}_t^Z + \mathcal{O}(\delta^{2H}) \\ &= \left[\delta^H \frac{1-\gamma}{q\gamma} \lambda(Z_0^{\delta,H}) \lambda'(Z_0^{\delta,H}) \frac{(T-t)^{H+\frac{1}{2}}}{\Gamma(H+\frac{3}{2})} \right] M_t d\widetilde{W}_t^Z + \mathcal{O}(\delta^{2H}), \end{aligned}$$

and the approximation of ξ_t is given by

$$\xi_t = \delta^H \frac{1-\gamma}{q\gamma} \lambda(Z_0^{\delta,H}) \lambda'(Z_0^{\delta,H}) \frac{(T-t)^{H+\frac{1}{2}}}{\Gamma(H+\frac{3}{2})} + \mathcal{O}(\delta^{2H}).$$

Plugging the above expression into (2.14) yields the desired result (3.19). \square

4 General Utilities and Fractional Stochastic Environment

In this section, we study the nonlinear portfolio optimization through asymptotics with general utility $U(x)$, and when the drift μ and volatility σ of the underlying asset S_t are driven slowly varying fractional stochastic factor $Z_t^{\delta,H}$ defined in (3.10). This is motivated by two recent works: in Fouque and Hu [2016a], we developed asymptotic results for the value function following a given strategy in the slow varying Markovian environment, and proved the optimality of such a strategy up to a certain order; on the other hand, asymptotics of linear pricing problem has been done and implied volatility is provided in Garnier and Solna [2015] when the volatility is driven by $Z_t^{\delta,H}$.

Using the notation $M(t, x; \lambda)$ for the classical Merton value with constant Sharpe-ratio λ , we denote by $v^{(0)}$ the value function at frozen Sharpe-ratio $\lambda(z)$,

$$v^{(0)}(t, x, z) = M(t, x, \lambda(z)). \quad (4.1)$$

Then we define the strategy $\pi^{(0)}$ by

$$\pi^{(0)}(t, x, z) = -\frac{\lambda(z) v_x^{(0)}(t, x, z)}{\sigma(z) v_{xx}^{(0)}(t, x, z)}, \quad (4.2)$$

and the associate value process $V^{\pi^{(0)}, \delta}$ is

$$V^{\pi^{(0)}, \delta} := \mathbb{E} \left[U \left(X_T^{\pi^{(0)}} \right) \middle| \mathcal{F}_t \right], \quad (4.3)$$

where $X_t^{\pi^{(0)}}$ is the wealth process following strategy $\pi^{(0)}$:

$$dX_t^{\pi^{(0)}} = \mu(Z_t^{\delta,H}) \pi^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta,H}) dt + \sigma(Z_t^{\delta,H}) \pi^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta,H}) dW_t. \quad (4.4)$$

We first derive the expansion for $V^{\pi^{(0)}, \delta}$, and then we show that $\pi^{(0)}$ is optimal up to order δ^H among the strategies of the form

$$\tilde{\mathcal{A}}_t^\delta[\tilde{\pi}^0, \tilde{\pi}^1, \alpha] := \left\{ \pi = \tilde{\pi}^0 + \delta^\alpha \tilde{\pi}^1 : \pi \in \mathcal{A}_t^\delta, \alpha > 0, 0 < \delta \leq 1 \right\}. \quad (4.5)$$

As a byproduct, by applying the expansion results for $V^{\pi^{(0)}, \delta}$ to power utility, $\pi^{(0)}$ obtained in Theorem 3.4 is optimal up to order δ^H within the full class of strategies \mathcal{A}_t^δ .

In the next subsection, we first review the classical Merton problem when μ and σ are constants in (2.1), which plays a crucial role in deriving the expansion (4.21) to $V^{\pi^{(0)}, \delta}$. Then we define some notations for later use.

4.1 Merton Problem with Constant Coefficients

This problem has been extensively studied, for example, in Karatzas and Shreve [1998]. Here we summarize the results about the classical Merton value function $M(t, x; \lambda)$.

Assume that the utility function $U(x)$ is $C^2(0, \infty)$, strictly increasing, strictly concave, and satisfies the Inada and Asymptotic Elasticity conditions:

$$U'(0+) = \infty, \quad U'(\infty) = 0, \quad \text{AE}[U] := \lim_{x \rightarrow \infty} x \frac{U'(x)}{U(x)} < 1,$$

then, the Merton value function $M(t, x; \lambda)$ is strictly increasing, strictly concave in the wealth variable x , and decreasing in the time variable t , which is $C^{1,2}([0, T] \times \mathbb{R}^+)$ and solves the HJB equation

$$M_t + \sup_{\pi} \left\{ \frac{1}{2} \sigma^2 \pi^2 M_{xx} + \mu \pi M_x \right\} = M_t - \frac{1}{2} \lambda^2 \frac{M_x^2}{M_{xx}} = 0, \quad M(T, x; \lambda) = U(x), \quad (4.6)$$

where $\lambda = \mu/\sigma$ is the constant Sharpe ratio. It is C^1 with respect to λ , and the optimal strategy is

$$\pi^*(t, x; \lambda) = -\frac{\lambda}{\sigma} \frac{M_x(t, x; \lambda)}{M_{xx}(t, x; \lambda)}. \quad (4.7)$$

Given the Merton value function $M(t, x; \lambda)$, one can define the risk-tolerance function by

$$R(t, x; \lambda) = -\frac{M_x(t, x; \lambda)}{M_{xx}(t, x; \lambda)}. \quad (4.8)$$

It is clear that $R(t, x; \lambda)$ is continuous and strictly positive due to the regularity, concavity and monotonicity of $M(t, x; \lambda)$. For further properties, we refer to Källblad and Zariphopoulou [2014, 2017] and Fouque and Hu [2016a]. We use the notation from Fouque et al. [2016]:

$$D_k = R(t, x; \lambda)^k \partial_x^k, \quad k = 1, 2, \dots, \quad (4.9)$$

$$\mathcal{L}_{t,x}(\lambda) = \partial_t + \frac{1}{2} \lambda^2 D_2 + \lambda^2 D_1. \quad (4.10)$$

Note that the coefficients of $\mathcal{L}_{t,x}(\lambda)$ depend on $R(t, x; \lambda)$, and therefore on $M(t, x; \lambda)$. The Merton PDE (4.6) can be re-written as

$$\mathcal{L}_{t,x}(\lambda)M(t, x; \lambda) = 0. \quad (4.11)$$

Next, we summarize all assumptions needed in the rest of this section. This will include properties of the utility function $U(x)$, the state processes $(X_t^{\pi^{(0)}}, S_t, Z_t^{\delta, H})$ as well as $v^{(0)}(t, x, z)$.

4.2 Assumptions

Basically, we work under the same set of assumptions as in Fouque and Hu [2016a], and we restate them here for readers' convenience. Detailed discussion about general utility functions can be found there in Section 2.3.

Assumption 4.1. *Throughout the paper, we make the following assumptions on the utility $U(x)$:*

- (i) $U(x)$ is $C^6(0, \infty)$, strictly increasing, strictly concave and satisfying the following conditions (Inada and Asymptotic Elasticity):

$$U'(0+) = \infty, \quad U'(\infty) = 0, \quad AE[U] := \lim_{x \rightarrow \infty} x \frac{U'(x)}{U(x)} < 1. \quad (4.12)$$

- (ii) $U(0+)$ is finite. Without loss of generality, we assume $U(0+) = 0$.

- (iii) Denote by $R(x)$ the risk tolerance,

$$R(x) := -\frac{U'(x)}{U''(x)}. \quad (4.13)$$

Assume that $R(0) = 0$, $R(x)$ is strictly increasing and $R'(x) < \infty$ on $[0, \infty)$, and there exists $K \in \mathbb{R}^+$, such that for $x \geq 0$, and $2 \leq i \leq 4$,

$$\left| \partial_x^{(i)} R^i(x) \right| \leq K. \quad (4.14)$$

- (iv) Define the inverse function of the marginal utility $U'(x)$ as $I : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $I(y) = U'^{-1}(y)$, and assume that, for some positive α, κ , $I(y)$ satisfies the polynomial growth condition:

$$I(y) \leq \alpha + \kappa y^{-\alpha}, \quad (4.15)$$

as well as for positive constants c_n, C_n , $n = 1, 2, 3$, with $c_2 > 1$,

$$c_1 I(x) \leq |xI'(x)| \leq C_1 I(x), \quad c_2 |I'(x)| \leq xI''(x) \leq C_2 |I'(x)| \quad \text{and} \quad |xI'''(x)| \leq C_3 I''(x), \quad (4.16)$$

Remark 4.2. The item (ii) excludes the case of power utility $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ when $\gamma > 1$. However, all results in this section still hold for the case $\gamma > 1$, with a slight modification in the proofs.

The conditions (4.16) which were introduced in Källblad and Zariphopoulou [2017], are crucial assumptions in their Proposition 4, which will be used in our derivation. They also give a mixture of inverse of the marginal utilities as an example that satisfies this conditions.

Below are the additional assumptions needed on the state processes $(X_t^{\pi^{(0)}}, S_t, Z_t)$ and on $v^{(0)}(t, x, z)$.

Assumption 4.3.

- (i) The function $\lambda(z) = \mu(z)/\sigma(z)$ is $C^2(\mathbb{R})$. Moreover, $\lambda(z)$, $\lambda'(z)$ and $\lambda''(z)$ are at most polynomially growing.
- (ii) The value function $v^{(0)}(t, x, z) = M(t, x; \lambda(z))$ satisfies the relation:

$$\left| x^2 v_{xx}^{(0)}(t, x, z) \right| \leq d(z) v^{(0)}(t, x, z), \quad (4.17)$$

with $d(z)$ being polynomial growth. Note that this is automatically satisfied by the power utility (2.5).

- (iii) The process $v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta, H})$ is in $L^4([0, T] \times \Omega)$ uniformly in δ , i.e.,

$$\mathbb{E} \left[\int_0^T \left(v^{(0)}(s, X_s^{\pi^{(0)}}, Z_0^{\delta, H}) \right)^4 ds \right] \leq C_1 \quad (4.18)$$

where C_1 is independent of δ and $Z_0^{\delta, H}$ is given in (3.10) with $t = 0$.

Remark 4.4. Notice that condition (4.17) is actually a hidden assumption on the general utility, and it is automatically satisfied by power utility. In order to guarantee (4.18), there is a list of assumptions discussed in [Fouque and Hu, 2016a, Section 2.4].

4.3 The Epsilon-Martingale Decomposition with a Given Strategy $\pi^{(0)}$

As introduced in Fouque et al. [2000] in the context of linear pricing problem and further developed in Garnier and Solna [2015], the idea of epsilon-martingale decomposition is to find a process which is in the form of a martingale plus something small with the right terminal condition. Specifically, we aim to find $Q^{\pi^{(0)}, \delta}$ such that its terminal condition coincides with the quantity of interest $V_t^{\pi^{(0)}, \delta}$, namely, $Q_T^{\pi^{(0)}, \delta} = V_T^{\pi^{(0)}, \delta} = U(X_T^{\pi^{(0)}})$, and that can be decomposed as

$$Q_t^{\pi^{(0)}, \delta} = M_t^\delta + R_t^\delta, \quad (4.19)$$

where M_t^δ is a martingale and R_t^δ is of order δ^{2H} . Note that the term of order δ^H will be absorbed in the martingale M_t^δ .

Suppose we obtain such a decomposition (4.19), and then taking conditional expectation with respect to \mathcal{F}_t on both sides of the equation $Q_T^{\pi^{(0)}, \delta} = M_T^\delta + R_T^\delta$ gives

$$V_t^{\pi^{(0)}, \delta} = \mathbb{E} \left[Q_T^{\pi^{(0)}, \delta} | \mathcal{F}_t \right] = M_t^\delta + \mathbb{E} \left[R_T^\delta | \mathcal{F}_t \right] = Q_t^{\pi^{(0)}, \delta} + \mathbb{E} \left[R_T^\delta | \mathcal{F}_t \right] - R_t^\delta. \quad (4.20)$$

Since R_t^δ is of order δ^{2H} , $Q_t^{\pi^{(0)}, \delta}$ is the approximation to $V_t^{\pi^{(0)}, \delta}$ up to δ^H . Therefore the above argument leads to the desired approximation result. Now it remains to find $Q_t^{\pi^{(0)}, \delta}$ so that the decomposition holds, and we have the following proposition.

Proposition 4.5. Under Assumption 4.1 and 4.3, for fixed $t \in [0, T)$, $X_t^{\pi^{(0)}} = x$, and the observed value $Z_0^{\delta, H}$, the \mathcal{F}_t -measurable value process $V_t^{\pi^{(0)}, \delta}$ defined in (4.3) is of the form

$$V_t^{\pi^{(0)}, \delta} = Q_t^{\pi^{(0)}, \delta}(X_t^{\pi^{(0)}}, Z_0^{\delta, H}) + \mathcal{O}(\delta^{2H}), \quad (4.21)$$

where $Q_t^{\pi^{(0)}, \delta}(x, z)$ is given by:

$$Q_t^{\pi^{(0)}, \delta}(x, z) = v^{(0)}(t, x, z) + \lambda(z) \lambda'(z) D_1 v^{(0)}(t, x, z) \phi_t^\delta + \delta^H \rho \lambda^2(z) \lambda'(z) v^{(1)}(t, x, z), \quad (4.22)$$

$v^{(0)}$ and D_1 are defined in (4.1) and (4.9) respectively, $(\phi_t^\delta)_{t \in [0, T]}$ is the \mathcal{F}_t -measurable process of order δ^H given in (3.16) and $v^{(1)}(t, x, z)$ is defined as

$$v^{(1)}(t, x, z) = D_1^2 v^{(0)}(t, x, z) D_{t, T}, \quad D_{t, T} = \frac{(T - t)^{H+3/2}}{\Gamma(H + \frac{5}{2})}. \quad (4.23)$$

The proof of Proposition 4.5 will be given after Corollary 4.6 and Proposition 4.7. As explained in Remark 3.5, we have the following corollary.

Corollary 4.6. *In the case of power utility $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, with $\gamma > 0$, $\gamma \neq 1$, and under Assumption 2.2 and 4.3, $\pi^{(0)}$ given by (3.19) is asymptotically optimal in the full class of admissible strategies \mathcal{A}_t^δ up to order δ^H .*

Proof. Straightforward computations give, under power utilities, $v^{(0)}$, $D_1 v^{(0)}$ and $v^{(1)}$ as

$$\begin{aligned} v^{(0)}(t, x, z) &= \frac{x^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma} \lambda^2(z)(T-t)}, \quad D_1 v^{(0)}(t, x, z) = \frac{x^{1-\gamma}}{\gamma} e^{\frac{1-\gamma}{2\gamma} \lambda^2(z)(T-t)}, \\ v^{(1)}(t, x, z) &= \frac{(1-\gamma)}{\gamma^2} x^{1-\gamma} e^{\frac{1-\gamma}{2\gamma} \lambda^2(z)(T-t)} \frac{(T-t)^{H+\frac{3}{2}}}{\Gamma(H+\frac{5}{2})}. \end{aligned}$$

Then, one can deduce $Q_t^\delta = Q_t^{\pi^{(0)}, \delta}$, where Q_t^δ is given by (3.15) and $Q_t^{\pi^{(0)}, \delta}$ is given by (4.22). Combining with Theorem 3.2, V^δ and $V^{\pi^{(0)}, \delta}$ admits the same first order approximation. Therefore, we obtain the desired asymptotic optimality. \square

For general utilities, we will derive a similar result in the smaller class $\tilde{\mathcal{A}}_t^\delta[\tilde{\pi}^0, \tilde{\pi}^1, \alpha]$ in Section 4.4.

Proposition 4.7. *For the Markovian case $H = \frac{1}{2}$, the approximation $Q_t^{\pi^{(0)}, \delta}$ given in (4.21) coincides with the result derived in [Fouque and Hu, 2016a, Theorem 3.1],*

$$V^{\pi^{(0)}, \delta}(t, x, z) = v^{(0)}(t, x, z) + \frac{\sqrt{\delta}}{2} (T-t)^2 \rho \lambda^2(z) \lambda'(z) D_1^2 v^{(0)}(t, x, z) + \mathcal{O}(\delta). \quad (4.24)$$

Proof. First observe that when $H = \frac{1}{2}$,

$$D_{t,T} = \frac{1}{2} (T-t)^2,$$

and the third term in $Q_t^{\pi^{(0)}, \delta}$ becomes $\frac{\sqrt{\delta}}{2} (T-t)^2 \rho \lambda^2(z) \lambda'(z) D_1^2 v^{(0)}(t, x, z)$. Using the moving-average representation (3.10) for $Z_s^{\delta, H}$ with $H = 1/2$, ϕ_t^δ is explicitly computed as

$$\phi_t^\delta = \frac{1 - e^{-a\delta(T-t)}}{a\delta} Z_t^{\delta, H} - (T-t) Z_0^{\delta, H} = (T-t) (Z_t^{\delta, H} - Z_0^{\delta, H}) + \mathcal{O}(\delta).$$

Then using the ‘‘Vega-Gamma’’ relation $v_z^{(0)}(t, x, z) = (T-t) \lambda(z) \lambda'(z) D_1 v^{(0)}(t, x, z)$ and the fact $(Z_t^{\delta, H} - Z_0^{\delta, H})^p \sim \mathcal{O}(\delta^{pH})$, one can deduce

$$\begin{aligned} V_t^{\pi^{(0)}, \delta} &= v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta, H}) + v_z^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta, H}) (Z_t^{\delta, H} - Z_0^{\delta, H}) \\ &\quad + \frac{\sqrt{\delta}}{2} (T-t)^2 \rho \lambda^2(Z_0^{\delta, H}) \lambda'(Z_0^{\delta, H}) D_1^2 v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta, H}) \\ &= v^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta, H}) + \frac{\sqrt{\delta}}{2} (T-t)^2 \rho \lambda^2(Z_t^{\delta, H}) \lambda'(Z_t^{\delta, H}) D_1^2 v^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta, H}) + \mathcal{O}(\delta), \end{aligned}$$

which is consistent with the result derived in [Fouque and Hu, 2016a, Theorem 3.1]. \square

We now turn to the proof of Proposition 4.5.

Proof of Proposition 4.5. According to the epsilon-martingale decomposition strategy, our goal is to show that $Q_t^{\pi^{(0)}, \delta}$ can be written as $M_t^\delta + R_t^\delta$, where M_t^δ is a martingale, and R_t^δ is of order δ^{2H} . We shall mainly focus on the derivation of $Q_t^{\pi^{(0)}, \delta}$ and delay the proofs of accuracy in the Appendix A for the sake of clarity and simplicity. The technique is very similar to the one presented in Garnier and Solna [2015] in the context of option pricing problem with fractional stochastic volatility. The main difference is that their case involves the linear Black-Scholes operator, as in our case, it involves the non-linear Merton operator

$\mathcal{L}_{t,x}(\lambda)$. Amazingly, the properties of risk-tolerance function $R(t, x; \lambda)$ will enable us to carry the proof as follows.

In order to avoid differentiating the fOU process $Z_t^{\delta,H}$, we freeze it at $Z_0^{\delta,H}$. The corresponding error will be compensated in the following calculation. This technique has also been use in the context of pricing when deriving hedging strategy with frozen volatility in [Fouque et al., 2011, Section 8.4].

By Itô's formula applied to $v^{(0)}$ defined in (4.1) and Taylor expansion in z at the point $Z_0^{\delta,H}$, we deduce

$$\begin{aligned}
dv^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) &= \mathcal{L}_{t,x}(\lambda(Z_t^{\delta,H}))v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) dt \\
&\quad + \sigma(Z_t^{\delta,H})\pi^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta,H})v_x^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) dW_t \\
&= \mathcal{L}_{t,x}(\lambda(Z_0^{\delta,H}))v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) dt \\
&\quad + \left[(Z_t^{\delta,H} - Z_0^{\delta,H})(\lambda^2 R)_z \Big|_{z=Z_0^{\delta,H}} + g_t^{(1)} \right] v_x^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) dt \\
&\quad + \left[(Z_t^{\delta,H} - Z_0^{\delta,H})(\lambda^2 R^2)_{zz} \Big|_{z=Z_0^{\delta,H}} + g_t^{(2)} \right] \frac{1}{2} v_{xx}^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) dt + dM_t^{(1)} \\
&= (Z_t^{\delta,H} - Z_0^{\delta,H})\lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H})D_1 v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) dt + dM_t^{(1)} \\
&\quad + g_t^{(1)} v_x^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) dt + \frac{1}{2} g_t^{(2)} v_{xx}^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) dt, \tag{4.25}
\end{aligned}$$

where in the derivation, we have used the relation

$$\mathcal{L}_{t,x}(\lambda(z))v^{(0)}(t, x, z) = 0, \quad D_1 v^{(0)} = -D_2 v^{(0)}, \quad \text{and} \quad \pi^{(0)}(t, x, z) = \frac{\lambda(z)}{\sigma(z)} R(t, x; \lambda(z)), \tag{4.26}$$

$M_t^{(1)}$ is the martingale defined by

$$dM_t^{(1)} = \sigma(Z_t^{\delta,H})\pi^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta,H})v_x^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) dW_t, \tag{4.27}$$

and the last two terms in (4.25) are of order δ^{2H} (see Appendix A), with $g_t^{(1)}$ and $g_t^{(2)}$ being the Lagrange remainders:

$$g_t^{(1)} = \frac{1}{2} \left(Z_t^{\delta,H} - Z_0^{\delta,H} \right)^2 (\lambda^2 R)_{zz} \Big|_{z=\chi_t^{(1)}}, \quad g_t^{(2)} = \frac{1}{2} \left(Z_t^{\delta,H} - Z_0^{\delta,H} \right)^2 (\lambda^2 R^2)_{zz} \Big|_{z=\chi_t^{(2)}}, \tag{4.28}$$

and $\chi_t^{(i)} \in [Z_0^{\delta,H} \wedge Z_t^{\delta,H}, Z_0^{\delta,H} \vee Z_t^{\delta,H}]$, $i = 1, 2$.

Now it remains to find the epsilon-martingale decomposition for the term $\int (Z_s^{\delta,H} - Z_0^{\delta,H}) D_1 v^{(0)}(s, X_s^{\pi^{(0)}}, Z_0^{\delta,H}) ds$ in (4.25). To this end, we recall ϕ_t^δ and ψ_t^δ given in (3.16) and (3.20) respectively, which satisfy the relation $(Z_t^{\delta,H} - Z_0^{\delta,H}) dt = d\psi_t^\delta - d\phi_t^\delta$ and consequently

$$(Z_t^{\delta,H} - Z_0^{\delta,H}) D_1 v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) dt = D_1 v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) (d\psi_t^\delta - d\phi_t^\delta). \tag{4.29}$$

On the right-hand side, the first term is proved to be a true martingale in Appendix A, while the second term need further analysis, namely, the differential of $\phi_t^\delta D_1 v^{(0)}$ will be computed. In the sequel, without any confusion, the arguments of $v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H})$ shall be omitted for simplicity.

$$\begin{aligned}
d(\phi_t^\delta D_1 v^{(0)}) &= D_1 v^{(0)} d\phi_t^\delta + \phi_t^\delta \mathcal{L}_{t,x}(\lambda(Z_t^{\delta,H})) D_1 v^{(0)} dt + \phi_t^\delta \sigma(Z_t^{\delta,H}) \pi^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta,H}) \partial_x D_1 v^{(0)} dW_t \\
&\quad + \sigma(Z_t^{\delta,H}) \pi^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta,H}) \partial_x D_1 v^{(0)} d\langle W, \phi^\delta \rangle_t \\
&= D_1 v^{(0)} d\phi_t^\delta + \rho \lambda(Z_0^{\delta,H}) D_1^2 v^{(0)} d\langle W^Z, \psi^\delta \rangle_t + \phi_t^\delta \sigma(Z_t^{\delta,H}) \pi^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta,H}) \partial_x D_1 v^{(0)} dW_t \\
&\quad + \phi_t^\delta g_t^{(3)} \partial_x D_1 v^{(0)} dt + \phi_t^\delta g_t^{(4)} \frac{1}{2} \partial_{xx} D_1 v^{(0)} dt + \rho g_t^{(5)} \partial_x D_1 v^{(0)} d\langle W^Z, \psi^\delta \rangle_t, \tag{4.30}
\end{aligned}$$

where in the above derivation, we have used

$$\mathcal{L}_{t,x}(\lambda(Z_0^{\delta,H})) D_1 v^{(0)} = D_1 \mathcal{L}_{t,x}(\lambda(Z_0^{\delta,H})) v^{(0)} = 0, \quad \text{and} \quad d\langle W, \phi^\delta \rangle_t = \rho d\langle W^Z, \psi^\delta \rangle_t, \tag{4.31}$$

with the first one being proved in [Fouque et al., 2016, Lemma 2.5]. Again, $g_t^{(3)}$, $g_t^{(4)}$ and $g_t^{(5)}$ are Lagrange remainders from Taylor series

$$\begin{aligned} g_t^{(3)} &= \left(Z_t^{\delta,H} - Z_0^{\delta,H} \right) (\lambda^2 R)_z \Big|_{z=\chi_t^{(3)}}, \quad g_t^{(4)} = \left(Z_t^{\delta,H} - Z_0^{\delta,H} \right) (\lambda^2 R^2)_z \Big|_{z=\chi_t^{(4)}}, \\ g_t^{(5)} &= \left(Z_t^{\delta,H} - Z_0^{\delta,H} \right) (\lambda R)_z \Big|_{z=\chi_t^{(5)}}, \end{aligned}$$

with $\chi_t^{(i)} \in \left[Z_0^{\delta,H} \wedge Z_t^{\delta,H}, Z_0^{\delta,H} \vee Z_t^{\delta,H} \right]$, $i = 3, 4, 5$.

Now combining (4.29) and (4.30) yields:

$$\begin{aligned} \left(Z_t^{\delta,H} - Z_0^{\delta,H} \right) D_1 v^{(0)} dt &= -d \left(\phi_t^\delta D_1 v^{(0)} \right) + \rho \lambda (Z_0^{\delta,H}) D_1^2 v^{(0)} d \langle W^z, \psi^\delta \rangle_t + dM_t^{(2)} \\ &\quad + \phi_t^\delta g_t^{(3)} \partial_x D_1 v^{(0)} dt + \phi_t^\delta g_t^{(4)} \frac{1}{2} \partial_{xx} D_1 v^{(0)} dt + \rho g_t^{(5)} \partial_x D_1 v^{(0)} d \langle W^Z, \psi^\delta \rangle_t \end{aligned}$$

where $M_t^{(2)}$ is the martingale given by

$$dM_t^{(2)} = D_1 v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) d\psi_t^\delta + \phi_t^\delta \sigma(Z_t^{\delta,H}) \pi^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta,H}) \partial_x D_1 v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) dW_t, \quad (4.32)$$

following a similar proof as for $M_t^{(1)}$.

Let $d \langle W^Z, \psi^\delta \rangle_t := \theta_{t,T}^\delta dt$, from Lemma A.1(iv), one has

$$\theta_{t,T}^\delta = \int_0^{T-t} \mathcal{K}^\delta(s) ds = \delta^H \theta_{t,T} + \delta^{H+1} \tilde{\theta}_{t,T},$$

and a straightforward computation gives

$$\partial_t D_{t,T} = -\theta_{t,T}, \quad (4.33)$$

where $D_{t,T}$ is defined in (4.23). Then applying Itô's formula to $v^{(1)}$ defined in (4.23) brings

$$\begin{aligned} dv^{(1)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) &= \mathcal{L}_{t,x}(\lambda(Z_t^{\delta,H})) v^{(1)} dt + \sigma(Z_t^{\delta,H}) \pi^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta,H}) v_x^{(1)} dW_t \\ &= \mathcal{L}_{t,x}(\lambda(Z_0^{\delta,H})) v^{(1)} dt + \sigma(Z_t^{\delta,H}) \pi^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta,H}) v_x^{(1)} dW_t \\ &\quad + g_t^{(3)} v_x^{(1)} dt + g_t^{(4)} \frac{1}{2} v_{xx}^{(1)} dt \\ &= -D_1^2 v^{(0)} \theta_{t,T} dt + dM_t^{(3)} + g_t^{(3)} v_x^{(1)} dt + g_t^{(4)} \frac{1}{2} v_{xx}^{(1)} dt, \end{aligned} \quad (4.34)$$

with the last two terms of order $\mathcal{O}(\delta^H)$, and $M_t^{(3)}$ as the martingale:

$$dM_t^{(3)} = \sigma(Z_t^{\delta,H}) \pi^{(0)}(t, X_t^{\pi^{(0)}}, Z_t^{\delta,H}) v_x^{(1)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) dW_t. \quad (4.35)$$

Collecting equation (4.25), (4.30) and (4.34), we obtain

$$\begin{aligned} dQ_t^{\pi^{(0)}, \delta}(X_t^{\pi^{(0)}}, Z_0^{\delta,H}) &= d \left(v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) + \lambda(Z_0^{\delta,H}) \lambda'(Z_0^{\delta,H}) \phi_t^\delta D_1 v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) \right. \\ &\quad \left. + \delta^H \rho \lambda^2(Z_0^{\delta,H}) \lambda'(Z_0^{\delta,H}) v^{(1)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta,H}) \right) \\ &= dM_t^\delta + dR_t^\delta, \end{aligned}$$

where dM_t^δ and dR_t^δ are

$$dM_t^\delta = dM_t^{(1)} + \lambda(Z_0^{\delta,H}) \lambda'(Z_0^{\delta,H}) dM_t^{(2)} + \delta^H \rho \lambda^2(Z_0^{\delta,H}) \lambda'(Z_0^{\delta,H}) dM_t^{(3)}, \quad (4.36)$$

$$dR_t^\delta = g_t^{(1)} v_x^{(0)} dt + \frac{1}{2} g_t^{(2)} v_{xx}^{(0)} dt + \delta^H \rho \lambda^2(Z_0^{\delta,H}) \lambda'(Z_0^{\delta,H}) \left[g_t^{(3)} v_x^{(1)} dt + g_t^{(4)} \frac{1}{2} v_{xx}^{(1)} dt \right] \quad (4.37)$$

$$+ \lambda(Z_0^{\delta,H}) \lambda'(Z_0^{\delta,H}) \left[\phi_t^\delta g_t^{(3)} \partial_x D_1 v^{(0)} dt + \phi_t^\delta g_t^{(4)} \frac{1}{2} \partial_{xx} D_1 v^{(0)} dt + \rho g_t^{(5)} \partial_x D_1 v^{(0)} \left(\delta^H \theta_{t,T} + \delta^{H+1} \tilde{\theta}_{t,T} \right) dt \right].$$

Noticing that $v^{(0)}(T, X_T^{\pi^{(0)}}, Z_0^{\delta, H}) = U(X_T^{\pi^{(0)}})$, $\phi_T^\delta D_1 v^{(0)}(T, X_T^{\pi^{(0)}}, Z_0^{\delta, H}) = 0$ since $\phi_T^\delta = 0$, and $v^{(1)}(T, X_T^{\pi^{(0)}}, Z_0^{\delta, H}) = 0$ by definition, the terminal condition for $Q^{\pi^{(0)}, \delta}$ indeed coincides with $V_T^{\pi^{(0)}, \delta}$. Combining with the proof that M_t^δ is a true martingale and R_t^δ is of order δ^{2H} detailed in Appendix A, we obtain the desired result in Proposition 4.5. \square

4.4 Asymptotic Optimality of $\pi^{(0)}$

In this subsection, we first derive the approximation of $V_t^{\pi, \delta}$

$$V_t^{\pi, \delta} := \mathbb{E}[U(X_T^\pi) | \mathcal{F}_t], \quad (4.38)$$

for any admissible strategy π taking the form $\tilde{\pi}^0 + \delta^\alpha \tilde{\pi}^1$ using epsilon-martingale decomposition technique as demonstrated in Proposition 4.5, where X_t^π is the wealth process following the trading strategy π :

$$dX_t^\pi = \mu(Z_t^{\delta, H})\pi(t, X_t^\pi, Z_t^{\delta, H}) dt + \sigma(Z_t^{\delta, H})\pi(t, X_t^\pi, Z_t^{\delta, H}) dW_t. \quad (4.39)$$

Then, given the previously established results of $V_t^{\pi^{(0)}, \delta}$ in Proposition 4.5, we asymptotically compare these approximations for $V_t^{\pi^{(0)}, \delta}$ and $V_t^{\pi, \delta}$, and then prove Theorem 4.8.

Theorem 4.8. *Under Assumptions 4.1, 4.3, 4.9 and B.1, for any family of trading strategies $\tilde{\mathcal{A}}_t^\delta[\tilde{\pi}^0, \tilde{\pi}^1, \alpha]$, the following limit exists in L^2 and satisfies*

$$\ell := \lim_{\delta \rightarrow 0} \frac{V_t^{\pi, \delta} - V_t^{\pi^{(0)}, \delta}}{\delta^H} \leq 0, \text{ in } L^2, \quad (4.40)$$

where $V_t^{\pi^{(0)}, \delta}$ and $V_t^{\pi, \delta}$ are defined in (4.3) and (4.38) respectively.

That is, the strategy $\pi^{(0)}$ that generate $V_t^{\pi^{(0)}, \delta}$ performs asymptotically better up to order δ^H than any family $\tilde{\mathcal{A}}_t^\delta[\tilde{\pi}^0, \tilde{\pi}^1, \alpha]$. Moreover, the inequality can be written according to the following four cases:

- (i) $\tilde{\pi}^0 = \pi^{(0)}$, $\alpha > H/2$: $\ell = 0$ and $V_t^{\pi, \delta} = V_t^{\pi^{(0)}, \delta} + o(\delta^H)$;
- (ii) $\tilde{\pi}^0 = \pi^{(0)}$, $\alpha = H/2$: $-\infty < \ell < 0$ and $V_t^{\pi, \delta} = V_t^{\pi^{(0)}, \delta} + \mathcal{O}(\delta^H)$ with $\mathcal{O}(\delta^H) < 0$;
- (iii) $\tilde{\pi}^0 = \pi^{(0)}$, $\alpha < H/2$: $\ell = -\infty$ and $V_t^{\pi, \delta} = V_t^{\pi^{(0)}, \delta} + \mathcal{O}(\delta^{2\alpha})$ with $\mathcal{O}(\delta^{2\alpha}) < 0$;
- (iv) $\tilde{\pi}^0 \neq \pi^{(0)}$: $\lim_{\delta \rightarrow 0} V_t^{\pi, \delta} < \lim_{\delta \rightarrow 0} V_t^{\pi^{(0)}, \delta}$,

where all relations between $V_t^{\pi, \delta}$ and $V_t^{\pi^{(0)}, \delta}$ hold under L^2 sense.

Assumption 4.9. *For a fixed choice of $(\tilde{\pi}^0, \tilde{\pi}^1, \alpha > 0)$, we require:*

- (i) *The whole family (in δ) of strategies $\{\tilde{\pi}^0 + \delta^\alpha \tilde{\pi}^1\}$ is contained in $\mathcal{A}^\delta(t, x, z)$;*
- (ii) *The function $\mu(z)$ is $C^1(\mathbb{R})$.*
- (iii) *Functions $\tilde{\pi}^0(t, x, z)$ and $\tilde{\pi}^1(t, x, z)$ are continuous on $[0, T] \times \mathbb{R}^+ \times \mathbb{R}$, and C^1 in z .*
- (iv) *The process $v^{(0)}(t, X_t^\pi, Z_0^{\delta, H})$ is in $L^4([0, T] \times \Omega)$ uniformly in δ , i.e.,*

$$\mathbb{E} \left[\int_0^T \left(v^{(0)}(s, X_s^\pi, Z_0^{\delta, H}) \right)^4 ds \right] \leq C_2 \quad (4.41)$$

where C_2 is independent of δ , $Z_0^{\delta, H}$ follows (3.10) with $t = 0$, and X_t^π follows (4.39) with $\pi = \tilde{\pi}^0 + \delta^\alpha \tilde{\pi}^1$.

Remark 4.10. *We have $\tilde{\pi}^0 + \delta^0 \tilde{\pi}^1 = \tilde{\pi}^0 + \tilde{\pi}^1 + \delta^\alpha \cdot 0$, so it is enough to consider $\alpha > 0$.*

Proof. We first deal with the case $\pi = \pi^{(0)} + \delta^\alpha \tilde{\pi}^1$. The derivation is similar to the one in Section 4.3. As usual, in order to condense the notation, we systematically omit the argument $(s, X_s^\pi, Z_0^{\delta, H})$ for $v^{(0)}$ in what follows.

$$\begin{aligned} dv^{(0)}(t, X_t^\pi, Z_0^{\delta, H}) &= v_t^{(0)} dt + \mu(Z_t^{\delta, H})\pi(t, X_t^\pi, Z_t^{\delta, H})v_x^{(0)} dt + \frac{1}{2}\sigma^2(Z_t^{\delta, H})\pi^2(t, X_t^\pi, Z_t^{\delta, H})v_{xx}^{(0)} dt \\ &\quad + \sigma(Z_t^{\delta, H})\pi(t, X_t^\pi, Z_t^{\delta, H})v_x^{(0)} dW_t \\ &= (Z_t^{\delta, H} - Z_0^{\delta, H})\lambda(Z_0^{\delta, H})\lambda'(Z_0^{\delta, H})D_1v^{(0)} dt + g_t^{(1)}v_x^{(0)} dt + \frac{1}{2}g_t^{(2)}v_{xx}^{(0)} dt + d\widetilde{M}_t^{(1)} \\ &\quad + \delta^\alpha \tilde{g}_t^{(1)}v_x^{(0)} dt + \delta^\alpha \tilde{g}_t^{(2)}v_{xx}^{(0)} dt + \frac{1}{2}\delta^{2\alpha}\sigma^2(Z_t^{\delta, H})(\tilde{\pi}^1)^2(t, X_t^\pi, Z_t^{\delta, H})v_{xx}^{(0)} dt, \end{aligned}$$

where $\widetilde{M}_t^{(1)}$, $\tilde{g}_t^{(1)}$ and $\tilde{g}_t^{(2)}$ are defined by

$$\begin{aligned} d\widetilde{M}_t^{(1)} &= \sigma(Z_t^{\delta, H})\left(\pi^{(0)}(t, X_t^\pi, Z_t^{\delta, H}) + \delta^\alpha \tilde{\pi}^1(t, X_t^\pi, Z_t^{\delta, H})\right)v_x^{(0)}(t, X_t^\pi, Z_0^{\delta, H})dW_t, \\ \tilde{g}_t^{(1)} &= \left(Z_t^{\delta, H} - Z_0^{\delta, H}\right)(\mu\tilde{\pi}^1)_z\Big|_{z=\tilde{\chi}_t^{(1)}}, \quad \tilde{g}_t^{(2)} = \left(Z_t^{\delta, H} - Z_0^{\delta, H}\right)(\mu R\tilde{\pi}^1)_z\Big|_{z=\tilde{\chi}_t^{(2)}}, \end{aligned}$$

with $\tilde{\chi}_t^{(i)} \in [Z_0^{\delta, H} \wedge Z_t^{\delta, H}, Z_0^{\delta, H} \vee Z_t^{\delta, H}]$, $i = 1, 2$.

Then it suffices to find the epsilon-martingale decomposition for the term $(Z_t^{\delta, H} - Z_0^{\delta, H})D_1v^{(0)}(t, X_t^\pi, Z_0^{\delta, H})dt$. Following a similar derivation as in Section 4.3, one can deduce

$$\begin{aligned} dQ_t^{\pi^{(0)}, \delta}(X_t^\pi, Z_0^{\delta, H}) &= d\left(v^{(0)}(t, X_t^\pi, Z_0^{\delta, H}) + \lambda(Z_0^{\delta, H})\lambda'(Z_0^{\delta, H})\phi_t^\delta D_1v^{(0)}(t, X_t^\pi, Z_0^{\delta, H})\right. \\ &\quad \left.+ \delta^H \rho \lambda^2(Z_0^{\delta, H})\lambda'(Z_0^{\delta, H})v^{(1)}(t, X_t^\pi, Z_0^{\delta, H})\right) \\ &= d\widetilde{M}_t^\delta + d\widetilde{R}_t^\delta + \delta^{2\alpha} dN_t^\delta, \end{aligned}$$

where

$$\begin{aligned} d\widetilde{M}_t^\delta &= d\widetilde{M}_t^{(1)} + \lambda(Z_0^{\delta, H})\lambda'(Z_0^{\delta, H})d\widetilde{M}_t^{(2)} + \delta^H \rho \lambda^2(Z_0^{\delta, H})\lambda'(Z_0^{\delta, H})d\widetilde{M}_t^{(3)}, \\ d\widetilde{M}_t^{(2)} &= D_1v^{(0)}(t, X_t^\pi, Z_0^{\delta, H})d\psi_t^\delta + \phi_t^\delta \sigma(Z_t^{\delta, H})\pi(t, X_t^\pi, Z_t^{\delta, H})\partial_x D_1v^{(0)}(t, X_t^\pi, Z_0^{\delta, H})dW_t, \\ d\widetilde{M}_t^{(3)} &= \sigma(Z_t^{\delta, H})\pi(t, X_t^\pi, Z_t^{\delta, H})v_x^{(1)}(t, X_t^\pi, Z_0^{\delta, H})dW_t, \\ d\widetilde{R}_t^\delta &= g_t^{(1)}v_x^{(0)} dt + \frac{1}{2}g_t^{(2)}v_{xx}^{(0)} dt + \delta^\alpha \tilde{g}_t^{(1)}v_x^{(0)} dt + \delta^\alpha \tilde{g}_t^{(2)}v_{xx}^{(0)} dt + \delta^H \rho \lambda^2(Z_0^{\delta, H})\lambda'(Z_0^{\delta, H})\left[g_t^{(3)}v_x^{(1)} + g_t^{(4)}\frac{1}{2}v_{xx}^{(1)}\right. \\ &\quad \left.+ \delta^\alpha \mu \tilde{\pi}^1 v_x^{(1)} + \delta^\alpha \sigma^2 \pi^{(0)} \tilde{\pi}^1 v_{xx}^{(1)} + \frac{1}{2}\delta^{2\alpha} \sigma^2 (\tilde{\pi}^1)^2 v_{xx}^{(1)}\right] dt + \lambda(Z_0^{\delta, H})\lambda'(Z_0^{\delta, H})\phi_t^\delta \left[g_t^{(3)}\partial_x D_1v^{(0)}\right. \\ &\quad \left.+ \frac{1}{2}g_t^{(4)}\partial_{xx} D_1v^{(0)} + \delta^\alpha \mu \tilde{\pi}^1 \partial_x D_1v^{(0)} + \delta^\alpha \sigma^2 \pi^{(0)} \tilde{\pi}^1 \partial_{xx} D_1v^{(0)} + \frac{1}{2}\delta^{2\alpha} \sigma^2 (\tilde{\pi}^1)^2 \partial_{xx} D_1v^{(0)}\right] dt \\ &\quad \left.+ \rho \lambda(Z_0^{\delta, H})\lambda'(Z_0^{\delta, H})\left[g_t^{(5)}\partial_x D_1v^{(0)}\left(\delta^H \theta_{t,T} + \delta^{H+1} \tilde{\theta}_{t,T}\right) + \delta^\alpha \sigma \tilde{\pi}^1 \partial_x D_1v^{(0)}\left(\delta^H \theta_{t,T} + \delta^{H+1} \tilde{\theta}_{t,T}\right)\right] dt, \\ d\widetilde{N}_t^\delta &= \frac{1}{2}\sigma^2(Z_t^{\delta, H})\left(\tilde{\pi}^1(t, X_t^\pi, Z_t^{\delta, H})\right)^2 v_{xx}^{(0)}(t, X_t^\pi, Z_0^{\delta, H})dt. \end{aligned}$$

To condense the expression for R_t^δ , we omit the arguments for functions $v^{(0)}(t, X_t^\pi, Z_0^{\delta, H})$, $v^{(1)}(t, X_t^\pi, Z_0^{\delta, H})$, $\mu(Z_t^{\delta, H})$, $\sigma(Z_t^{\delta, H})$, $\pi^{(0)}(t, X_t^\pi, Z_t^{\delta, H})$ and $\tilde{\pi}^1(t, X_t^\pi, Z_t^{\delta, H})$.

Since the Merton value $M(t, x; \lambda)$ is strictly concave, so does $v^{(0)}(t, x, z) = M(t, x; \lambda(z))$, which implies that N_t is non-increasing. Moreover, under Assumption 4.9, B.1, one can prove \widetilde{M}_t^δ is a true martingale

and \tilde{R}_t^δ is of order $\delta^{H+H\wedge\alpha}$, which yields

$$\begin{aligned} V_t^{\pi,\delta} &= \mathbb{E} \left[Q_T^{\pi^{(0)},\delta} | \mathcal{F}_t \right] = \tilde{M}_t^\delta + \mathbb{E} \left[\tilde{R}_T^\delta + \delta^{2\alpha} N_T^\delta | \mathcal{F}_t \right] \\ &= Q_t^{\pi^{(0)},\delta}(X_t^\pi, Z_0^{\delta,H}) + \mathbb{E} \left[\tilde{R}_T^\delta - \tilde{R}_t^\delta | \mathcal{F}_t \right] + \delta^{2\alpha} \mathbb{E} \left[N_T^\delta - N_t^\delta | \mathcal{F}_t \right] \\ &= Q_t^{\pi^{(0)},\delta}(X_t^\pi, Z_0^{\delta,H}) + \delta^{2\alpha} \mathbb{E} \left[N_T^\delta - N_t^\delta | \mathcal{F}_t \right] + \mathcal{O}(\delta^{H+H\wedge\alpha}) \leq Q_t^{\pi^{(0)},\delta}(X_t^\pi, Z_0^{\delta,H}) + \mathcal{O}(\delta^{H+H\wedge\alpha}), \end{aligned} \quad (4.42)$$

where in the derivation we have used $\tilde{M}_t^\delta + \tilde{R}_t^\delta + N_t^\delta = Q_t^{\pi^{(0)},\delta}(X_t^\pi, Z_0^{\delta,H})$ and the decreasing property of N_t .

The second case is $\pi = \tilde{\pi}^0 + \delta^\alpha \tilde{\pi}^1$ with $\tilde{\pi}^0 \neq \pi^{(0)}$. Here the wealth process X_t^π follows

$$dX_t^\pi = \mu(Z_t^{\delta,H}) (\tilde{\pi}^0 + \delta^\alpha \tilde{\pi}^1) (t, X_t^\pi, Z_t^{\delta,H}) dt + \sigma(Z_t^{\delta,H}) (\tilde{\pi}^0 + \delta^\alpha \tilde{\pi}^1) (t, X_t^\pi, Z_t^{\delta,H}) dW_t. \quad (4.43)$$

Under similar derivations, one can deduce

$$dv^{(0)}(t, X_t^\pi, Z_0^{\delta,H}) = d\hat{M}_t^\delta + d\hat{R}_t^\delta + d\hat{N}_t^\delta \quad (4.44)$$

where

$$d\hat{M}_t^\delta = \sigma(Z_t^{\delta,H}) \pi(t, X_t^\pi, Z_t^{\delta,H}) v_x^{(0)}(t, X_t^\pi, Z_0^{\delta,H}) dW_t, \quad (4.45)$$

$$d\hat{R}_t^\delta = \left[\hat{g}_t^{(1)} v_x^{(0)} + \frac{1}{2} \hat{g}_t^{(2)} v_{xx}^{(0)} \right] dt + \delta^\alpha \left[\mu \tilde{\pi}^1 v_x^{(0)} + \sigma^2 \tilde{\pi}^0 \tilde{\pi}^1 v_{xx}^{(0)} + \frac{1}{2} \delta^\alpha \sigma^2 (\tilde{\pi}^1)^2 v_{xx}^{(0)} \right] dt, \quad (4.46)$$

$$d\hat{N}_t^\delta = \frac{1}{2} \sigma^2(Z_0^{\delta,H}) (\tilde{\pi}^0 - \pi^{(0)})^2 (t, X_t^\pi, Z_0^{\delta,H}) v_{xx}^{(0)}(t, X_t^\pi, Z_0^{\delta,H}) dt, \quad (4.47)$$

with $\hat{g}_t^{(1)}$ and $\hat{g}_t^{(2)}$ defined as

$$\hat{g}_t^{(1)} = \left(Z_t^{\delta,H} - Z_0^{\delta,H} \right) (\mu \tilde{\pi}^0)_z \Big|_{z=\hat{\chi}_t^{(1)}}, \quad \hat{g}_t^{(2)} = \left(Z_t^{\delta,H} - Z_0^{\delta,H} \right) (\sigma^2 (\tilde{\pi}^0)^2)_z \Big|_{z=\hat{\chi}_t^{(2)}}, \quad (4.48)$$

and $\hat{\chi}_t^{(i)} \in [Z_0^{\delta,H} \wedge Z_t^{\delta,H}, Z_0^{\delta,H} \vee Z_t^{\delta,H}]$, $i = 1, 2$.

Here \hat{N}_t^δ is strictly decreasing due to the strict concavity of $v^{(0)}$. Under Assumption 4.9, B.1, \hat{M}_t^δ is a true martingale, and \hat{R}_t^δ is of order $\delta^{H\wedge\alpha}$. Therefore we obtain

$$V_t^{\pi,\delta} = v^{(0)}(t, X_t^\pi, Z_0^{\delta,H}) + \mathbb{E} \left[\hat{N}_T^\delta - \hat{N}_t^\delta | \mathcal{F}_t \right] + \mathcal{O}(\delta^{H\wedge\alpha}) < v^{(0)}(t, X_t^\pi, Z_0^{\delta,H}) + \mathcal{O}(\delta^{H\wedge\alpha}). \quad (4.49)$$

Now comparing the approximation (4.21) with (4.42) (4.49), we obtain the desired result in Theorem 4.8. \square

5 Conclusion

In this paper, we have considered the portfolio allocation problem in the context of a slowly varying fractional stochastic environment driven by a fractional OU process with $H \in (0, 1)$, and when the investor tries to maximize her terminal utility with, first, power utilities, and, then, in a general class of utility functions.

In the power utility case, using a martingale distortion representation for the value process and the epsilon-martingale decomposition method, we are able to derive a first order asymptotic approximation for both the optimal portfolio value and the optimal strategy. The first order correction for the optimal portfolio value has both random and deterministic parts as in the linear option pricing problem studied in Garnier and Solna [2015]. However, the approximate optimal strategy does not involve a random part and can be easily implemented. We also show that the zeroth order of the optimal strategy generates the

portfolio value up to the first order and we observe that the first order correction is even more important in the case of H small as observed in volatility data (see Gatheral et al. [2014]).

Finally, we extend our analysis to the case of general utilities where we can derive the first order asymptotic optimality within a specific subclass of strategies.

The case of fast varying fractional stochastic environment with $H \in (\frac{1}{2}, 1)$ is the topic of the paper in preparation Fouque and Hu [2017].

A Technical Lemmas

In this section, we present several lemmas which are used in Section 3 and 4.

Lemma A.1. (i) *The slowly varying fractional factor $Z_t^{\delta, H}$ defined in (3.10) is a stationary Gaussian process with zero mean and variance*

$$\mathbb{E} \left[\left(Z_t^{\delta, H} \right)^2 \right] = \int_{-\infty}^t (\mathcal{K}^\delta(t-s))^2 ds = \int_0^\infty \mathcal{K}^2(s) ds = \sigma_{ou}^2, \quad (\text{A.1})$$

where σ_{ou}^2 is given in (3.6) and free of δ . Therefore $Z_t^{\delta, H}$ has finite moments of any order, and for any $p \in \mathbb{N}^+$, $Z_t^{\delta, H} \in L^p([0, T] \times \Omega)$ uniformly in δ .

Any adapted process that $\chi_t \in [Z_0^{\delta, H} \wedge Z_t^{\delta, H}, Z_0^{\delta, H} \vee Z_t^{\delta, H}]$ also satisfies that $\chi_\cdot \in L^p([0, T] \times \Omega)$ uniformly in δ .

(ii) *The difference $Z_t^{\delta, H} - Z_0^{\delta, H}$ is a Gaussian random variable with zero mean and variance*

$$\mathbb{E} \left[\left(Z_t^{\delta, H} - Z_0^{\delta, H} \right)^2 \right] = \sigma_H^2 (\delta t)^{2H} + o(\delta^{2H}), \quad (\text{A.2})$$

where σ_H^2 is given in (3.3). Consequently, the k^{th} moment of $Z_t^{\delta, H} - Z_0^{\delta, H}$ is of order δ^{kH} , uniformly in $t \in [0, T]$. Moreover, $Z_t^{\delta, H} - Z_0^{\delta, H}$ is of order δ^H in $L^p([0, T] \times \Omega)$ sense, for any $p \in \mathbb{N}^+$.

(iii) *The random correction ϕ_t^δ defined in (3.16) is a normal random variable of order δ^H with zero mean and variance*

$$\begin{aligned} \mathbb{E} \left[(\phi_t^\delta)^2 \right] &= \frac{\delta^{2H} T^{2+2H}}{\Gamma^2(H + \frac{3}{2})} \int_0^\infty \left[\left(1 - \frac{t}{T} + v \right)^{H+\frac{1}{2}} - v^{H+\frac{1}{2}} - (1 - \frac{t}{T})(H + \frac{1}{2})(v - \frac{t}{T})_+^{H-\frac{1}{2}} \right]^2 dv \\ &\quad + \mathcal{O}(\delta^{2H+1}), \end{aligned} \quad (\text{A.3})$$

where the integral is uniformly bounded in $t \in [0, T]$. Therefore, the $L^p([0, T] \times \Omega)$ norm ϕ_t^δ is of order δ^H , for any $p \in \mathbb{N}^+$.

(iv) *The process $(\psi_t^\delta)_{t \in [0, T]}$ defined in (3.20) is a square-integrable martingale satisfying*

$$d\psi_t^\delta = \int_0^{T-t} \mathcal{K}^\delta(s-t) ds dW_t^Z := \left(\delta^H \theta_{t,T} + \delta^{H+1} \tilde{\theta}_{t,T} \right) dW_t^Z, \quad (\text{A.4})$$

with $\theta_{t,T}$ and $\tilde{\theta}_{t,T}$ given by

$$\theta_{t,T} = \frac{1}{\Gamma(H + \frac{3}{2})} (T-t)^{H+\frac{1}{2}}, \quad \tilde{\theta}_{t,T} = \frac{a}{\Gamma(H + \frac{1}{2})} \int_0^{T-t} \int_0^s (s-u)^{H-\frac{1}{2}} e^{-a\delta u} du ds \leq \frac{a(T-t)^{H+\frac{3}{2}}}{\Gamma(H + \frac{5}{2})},$$

and uniformly bounded in $t \in [0, T]$ and $\delta \ll 1$. Consequently, one has

$$d \langle \psi, W^Z \rangle_t = \left(\int_0^{T-t} \mathcal{K}^\delta(s) ds \right) dt \quad \text{and} \quad d \langle \psi \rangle_t = \left(\int_0^{T-t} \mathcal{K}^\delta(s) ds \right)^2 dt. \quad (\text{A.5})$$

Proof. All can be computed directly, and we refer to the statements in [Garnier and Solna, 2015, Section 6, Appendix A]. \square

Lemma A.2. *The processes $(M^{(i)})_{t \in [0, T]}$, $i = 1, 2, 3$ defined in (4.27), (4.32) and (4.35) are true martingales with respect to the filtration \mathcal{F}_t , so does $(M)_{t \in [0, T]}$.*

Proof. We prove this result by showing $\mathbb{E} \left[\langle M^{(1)} \rangle_T^{1/2} \right] < \infty$, which is equivalent to $\mathbb{E} \left[\sup_{s \leq T} |M_s^{(1)}| \right] < \infty$ by Burkholder–Davis–Gundy inequality. This implies that $M^{(1)}$ is a martingale.

To this end, we first bound its quadratic variation

$$\begin{aligned} d \langle M^{(1)} \rangle_t &= \lambda^2(Z_t^{\delta, H}) R^2(t, X_t^{\pi^{(0)}}; \lambda(Z_t^{\delta, H})) \left(v_x^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta, H}) \right)^2 dt \\ &\leq \lambda^2(Z_t^{\delta, H}) C^2 \left(X_t^{\pi^{(0)}} v_x^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta, H}) \right)^2 dt \leq \lambda^2(Z_t^{\delta, H}) C^2 \left(v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta, H}) \right)^2 dt \end{aligned}$$

by using the estimate $R(t, x; \lambda(z)) \leq Cx$ and the concavity of $v^{(0)}$, and then deduce

$$\begin{aligned} \mathbb{E} \left[\langle M^{(1)} \rangle_T^{1/2} \right] &\leq C^2 \mathbb{E} \left[\left(\int_0^T \lambda^2(Z_s^{\delta, H}) \left(v^{(0)}(s, X_s^{\pi^{(0)}}, Z_0^{\delta, H}) \right)^2 ds \right)^{1/2} \right] \\ &\leq C^2 \mathbb{E}^{1/4} \left[\int_0^T \lambda^4(Z_s^{\delta, H}) ds \right] \cdot \mathbb{E}^{1/4} \left[\int_0^T \left(v^{(0)}(s, X_s^{\pi^{(0)}}, Z_0^{\delta, H}) \right)^4 ds \right] < \infty, \end{aligned}$$

where to conclude, we have used Assumption 4.3, and Lemma A.1(i) about $Z_s^{\delta, H}$.

The proofs for $M^{(2)}$ and $M^{(3)}$ are obtained in a similar way with estimates from [Fouque and Hu, 2016a, Proposition 3.5], which is of the form

$$\left| R^j(t, x; \lambda(z)) \left(\partial_x^{(j+1)} R(t, x; \lambda(z)) \right) \right| \leq K_j, \quad \forall 0 \leq j \leq 3, \quad \forall (t, x, z) \in [0, T) \times \mathbb{R}^+ \times \mathbb{R}, \quad (\text{A.6})$$

and Lemma A.1(iii)-(iv), and thus we omit the details here. \square

Lemma A.3. *The process $(R_t^\delta)_{t \in [0, T]}$ defined in (4.37) is of order δ^{2H} .*

Proof. We shall prove that each term in R_t^δ is of order δ^{2H} . The first term we deal with is $g_t^{(1)} v_x^{(0)}$ with $g_t^{(1)}$ defined in (4.28):

$$\begin{aligned} \left| g_t^{(1)} v_x^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta, H}) \right| &= \frac{1}{2} \left(Z_t^{\delta, H} - Z_0^{\delta, H} \right)^2 \left| 2(\lambda')^2 R + 2\lambda\lambda'' R + 4\lambda\lambda' R_z + \lambda^2 R_{zz} \right|_{z=\chi_t^{(1)}} v_x^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta, H}) \\ &\leq \frac{1}{2} \left(Z_t^{\delta, H} - Z_0^{\delta, H} \right)^2 d(\chi_t^{(1)}) R(t, X_t^{\pi^{(0)}}; \lambda(\chi_t^{(1)})) v_x^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta, H}) \\ &\leq \frac{1}{2} \left(Z_t^{\delta, H} - Z_0^{\delta, H} \right)^2 d(\chi_t^{(1)}) C X_t^{\pi^{(0)}} v_x^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta, H}) \\ &\leq C \left(Z_t^{\delta, H} - Z_0^{\delta, H} \right)^2 d(\chi_t^{(1)}) v^{(0)}(t, X_t^{\pi^{(0)}}, Z_0^{\delta, H}). \end{aligned}$$

Here the first inequality follows from [Fouque and Hu, 2016a, Proposition 3.7]: there exists non-negative functions $\tilde{d}_{01}(z)$ and $\tilde{d}_{02}(z)$ that have mostly polynomial growth and satisfy

$$|R_z(t, x; \lambda(z))| \leq \tilde{d}_{01}(z) R(t, x; \lambda(z)), \quad |R_{zz}(t, x; \lambda(z))| \leq \tilde{d}_{02}(z) R(t, x; \lambda(z)),$$

and thus $d(z)$ is also at most polynomially growing defined as

$$d(z) = \left| 2(\lambda'(z))^2 + 2\lambda(z)\lambda''(z) + 4\lambda(z)\lambda'(z)\tilde{d}_{01}(z) + \lambda^2(z)\tilde{d}_{02}(z) \right|. \quad (\text{A.7})$$

The second inequality is given by the estimate $R(t, x; \lambda(z)) \leq Cx$ and the concavity of $v^{(0)}$. Therefore

$$\begin{aligned} \mathbb{E} \left[\int_0^T g_s^{(1)} v_x^{(0)}(s, X_s^{\pi^{(0)}}, Z_0^{\delta, H}) ds \right] &\leq C \mathbb{E} \left[\int_0^T \left(Z_s^{\delta, H} - Z_0^{\delta, H} \right)^2 d(\chi_s^{(1)}) v^{(0)}(s, X_s^{\pi^{(0)}}, Z_0^{\delta, H}) ds \right] \\ &\leq \left[\mathbb{E} \int_0^T \left(Z_s^{\delta, H} - Z_0^{\delta, H} \right)^8 ds \right]^{\frac{1}{4}} \left[\mathbb{E} \int_0^T d^4(\chi_s^{(1)}) ds \right]^{\frac{1}{4}} \left[\mathbb{E} \int_0^T \left(v^{(0)}(s, X_s^{\pi^{(0)}}, Z_0^{\delta, H}) \right)^2 ds \right]^{\frac{1}{2}} \end{aligned}$$

and is of order δ^{2H} . This is because, one has proved in Lemma A.1(ii) that the first expectation is of order δ^{2H} , the second expectation is uniformly bounded in δ due to the polynomial growth property of $d(\cdot)$ and Lemma A.1(i), while the third term is uniformly bounded by Assumption 4.3(iii).

Other terms contained in R_t^δ can be proved of order δ^{2H} in a similar way with additional Assumption 4.3(ii), estimates (A.6), Lemma A.1(iii)-(iv) and estimates from [Källblad and Zariphopoulou, 2017, Proposition 4]. \square

B Assumptions in Section 4.4

This set of assumptions is used in establishing the approximation accuracy (4.42) (resp. (4.49)) to V_t^π defined in (4.38), namely, these assumptions will ensure that \widehat{M}_t^δ (resp. \widehat{M}_t^δ) is a true martingale and that \widetilde{R}_t^δ (resp. \widehat{R}_t^δ) is of order $\delta^{H+H\wedge\alpha}$ (resp. $\delta^{H\wedge\alpha}$).

Assumption B.1. Let $\mathcal{A}_0(t, x, z) [\widetilde{\pi}^0, \widetilde{\pi}^1, \alpha]$ be the family of trading strategies defined in (4.5). Recall that X^π is the wealth generated by the strategy $\pi = \widetilde{\pi}^0 + \delta^\alpha \widetilde{\pi}^1$ as defined in (4.39). In order to condense the notation, we systematically omit the argument $(s, X_s^\pi, Z_0^{\delta, H})$ of $v^{(0)}$ and $v^{(1)}$, the argument $Z_s^{\delta, H}$ of μ and σ , the argument $Z_0^{\delta, H}$ of λ , and $(s, X_s^\pi, Z_s^{\delta, H})$ of $\widetilde{\pi}^0$ and $\widetilde{\pi}^1$ in what follows. According to the different cases, we further require:

(i) If $\widetilde{\pi}^0 \equiv \pi^{(0)}$, the following quantities are uniformly bounded in δ :

$$\begin{aligned} &\mathbb{E} \int_0^T \left((\mu \widetilde{\pi}^1)_z|_{z=\widetilde{\xi}_s^{(1)}} v_x^{(0)} \right)^2 ds, \mathbb{E} \int_0^T \left((\mu R \widetilde{\pi}^1)_z|_{z=\widetilde{\xi}_s^{(2)}} v_{xx}^{(0)} \right)^2 ds, \mathbb{E} \int_0^T \left(\mu \widetilde{\pi}^1 v_x^{(0)} \right)^2 ds, \mathbb{E} \int_0^T \left(\sigma \widetilde{\pi}^1 v_x^{(0)} \right)^2 ds, \\ &\mathbb{E} \int_0^T \left(\sigma^2 (\widetilde{\pi}^1)^2 \partial_{xx} D_1 v^{(0)} \right)^2 ds, \mathbb{E} \left[\lambda^2 \lambda' \int_0^T \mu \widetilde{\pi}^1 v_x^{(1)} ds \right], \mathbb{E} \left[\lambda^2 \lambda' \int_0^T \sigma^2 (\widetilde{\pi}^1)^2 v_{xx}^{(1)} ds \right], \\ &\mathbb{E} \left[\lambda \lambda' \left(\int_0^T \left(\sigma \widetilde{\pi}^1 v_x^{(0)} \phi_s^\delta \right)^2 ds \right)^{\frac{1}{2}} \right], \mathbb{E} \left[\lambda^2 \lambda' \left(\int_0^T \left(\sigma \widetilde{\pi}^1 v_x^{(1)} \right)^2 ds \right)^{\frac{1}{2}} \right], \end{aligned}$$

(ii) If $\widetilde{\pi}^0 \neq \pi^{(0)}$, we require the uniformly boundedness (in δ) of the following:

$$\begin{aligned} &\mathbb{E} \int_0^T \left((\mu \widetilde{\pi}^0)_z|_{z=\widetilde{\xi}_s^{(1)}} v_x^{(0)} \right)^2 ds, \mathbb{E} \int_0^T \left((\sigma^2 (\widetilde{\pi}^0)^2)_z|_{z=\widetilde{\xi}_s^{(2)}} v_{xx}^{(0)} \right)^2 ds, \mathbb{E} \int_0^T \mu \widetilde{\pi}^1 v_x^{(0)} ds, \mathbb{E} \int_0^T \sigma^2 \widetilde{\pi}^0 \widetilde{\pi}^1 v_{xx}^{(0)} ds, \\ &\mathbb{E} \int_0^T \sigma^2 (\widetilde{\pi}^1)^2 v_{xx}^{(0)} ds, \mathbb{E} \left(\int_0^T \left(\sigma \widetilde{\pi}^0 v_x^{(1)} \right)^2 ds \right)^{\frac{1}{2}}, \mathbb{E} \left(\int_0^T \left(\sigma \widetilde{\pi}^1 v_x^{(1)} \right)^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

References

- F. Biagini, Y. Hu, B. Øksendal, and T. Zhang. *Stochastic calculus for fractional Brownian motion and applications*. Springer Science & Business Media, 2008.
- G. Chacko and L. M. Viceira. Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. *Review of Financial Studies*, 18(4):1369–1402, 2005.
- P. Cheridito, H. Kawaguchi, and M. Maejima. Fractional ornstein-uhlenbeck processes. *Electronic Journal of Probability*, 8(3):1–14, 2003.

- L. Coutin. An introduction to (stochastic) calculus with respect to fractional brownian motion. In *Séminaire de Probabilités XL*, pages 3–65. Springer, 2007.
- J. C. Cox and C. Huang. Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of economic theory*, 49:33–83, 1989.
- J. Cvitanic and I. Karatzas. On portfolio optimization under "drawdown" constraints. *IMA volumes in mathematics and its applications*, 65:35–35, 1995.
- G. Di Nunno, B. K. Øksendal, and F. Proske. *Malliavin calculus for Lévy processes with applications to finance*, volume 2. Springer, 2009.
- R. Elie and N. Touzi. Optimal lifetime consumption and investment under a drawdown constraint. *Finance and Stochastics*, 12:299–330, 2008.
- J.-P. Fouque and R. Hu. Asymptotic optimal strategy for portfolio optimization in a slowly varying stochastic environment. *arXiv preprint arXiv:1603.03538*, 2016a.
- J.-P. Fouque and R. Hu. Asymptotic methods for portfolio optimization problem in multiscale stochastic environments, 2016b. In preparation.
- J.-P. Fouque and R. Hu. Optimal portfolio under fast mean-reverting fractional stochastic environment, 2017. In preparation.
- J.-P. Fouque, G. Papanicolaou, and R. Sircar. *Derivatives in financial markets with stochastic volatility*. Cambridge University Press, 2000.
- J.-P. Fouque, G. Papanicolaou, and R. Sircar. Stochastic volatility and epsilon-martingale decomposition. In *Trends in Mathematics, Birkhauser Proceedings of the Workshop on Mathematical Finance.*, pages 152–161. Springer, 2001.
- J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna. *Multiscale Stochastic Volatility for Equity, Interest-Rate and Credit Derivatives*. Cambridge University Press, 2011.
- J.-P. Fouque, R. Sircar, and T. Zariphopoulou. Portfolio optimization & stochastic volatility asymptotics. *Mathematical Finance*, 2016.
- J. Garnier and K. Solna. Correction to black-scholes formula due to fractional stochastic volatility. *arXiv preprint arXiv:1509.01175*, 2015.
- J. Gatheral, T. Jaisson, and M. Rosenbaum. Volatility is rough. *arXiv preprint arXiv:1410.3394*, 2014.
- S. J. Grossman and Z. Zhou. Optimal investment strategies for controlling drawdowns. *Mathematical Finance*, 3:241–276, 1993.
- P. Guasoni and J. Muhle-Karbe. Portfolio choice with transaction costs: a users guide. In *Paris-Princeton Lectures on Mathematical Finance 2013*, pages 169–201. Springer, 2013.
- T. Jaisson and M. Rosenbaum. Rough fractional diffusions as scaling limits of nearly unstable heavy tailed hawkes processes. *The Annals of Applied Probability*, 26(5):2860–2882, 2016.
- T. Kaarakka and P. Salminen. On fractional ornstein-uhlenbeck process. *Communications on Stochastic Analysis*, 5(1):121–133, 2011.
- S. Källblad and T. Zariphopoulou. Qualitative analysis of optimal investment strategies in log-normal markets. *Available at SSRN 2373587*, 2014.
- S. Källblad and T. Zariphopoulou. On the black's equation for the local risk tolerance function. *Preprints*, 2017.
- I. Karatzas and S. E. Shreve. *Methods of Mathematical Finance*. Springer Science & Business Media, 1998.

- I. Karatzas, J. P. Lehoczky, and S. E. Shreve. Optimal portfolio and consumption decisions for a “small investor” on a finite horizon. *SIAM journal on control and optimization*, 25(6):1557–1586, 1987.
- D. Kramkov and W. Schachermayer. Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Annals of Applied Probability*, pages 1504–1516, 2003.
- M. Lorig and R. Sircar. Portfolio optimization under local-stochastic volatility: Coefficient taylor series approximations and implied sharpe ratio. *SIAM Journal on Financial Mathematics*, 7(1):418–447, 2016.
- M. J. Magill and G. M. Constantinides. Portfolio selection with transactions costs. *Journal of Economic Theory*, 13:245–263, 1976.
- B. B. Mandelbrot and J. W. Van Ness. Fractional brownian motions, fractional noises and applications. *SIAM review*, 10(4):422–437, 1968.
- R. C. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and statistics*, 51:247–257, 1969.
- R. C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of economic theory*, 3(4):373–413, 1971.
- D. Nualart. Malliavin calculus and its applications, volume 110 of cbms regional conference series in mathematics. In *Published for the Conference Board of the Mathematical Sciences, Washington, DC*, 2009.
- E. E. Omar, F. Masaaki, and M. Rosenbaum. The microstructural foundations of leverage effect and rough volatility. *arXiv preprint arXiv:1609.05177*, 2016.
- M. Tehranchi. Explicit solutions of some utility maximization problems in incomplete markets. *Stochastic Processes and their Applications*, 114(1):109–125, 2004.
- T. Zariphopoulou. Optimal investment and consumption models with non-linear stock dynamics. *Mathematical Methods of Operations Research*, 50(2):271–296, 1999.